Polynomial solutions of algebraic difference equations and homogeneous symmetric polynomials

Technical Report

Olha Shkaravska and Marko van Eekelen

Radboud University, Nijmegen and Open University of the Netherlands, Heerlen

Abstract

This article addresses the problem of computing an upper bound of the degree d of a polynomial solution $P(x)$ of an algebraic difference equation of the form $G\big(x\big)\big(P(x-\tau_1),\ldots,P(x-\tau_s)\big)+G_0(x)=0$ when such $P(x)$ with the coefficients in a field $\mathbb{K}$ of characteristic zero exists and where G is a non-linear s-variable polynomial with coefficients in $\mathbb{K}[x]$ and G_0 is a polynomial with coefficients in $\mathbb{K}$.

It will be shown that if G is a quadratic polynomial with constant coefficients then one can construct a countable family of polynomials $f_l(u_0)$ such that if there exists a (minimal) index l_0 with $f_{l_0}(u_0)$ being a non-zero polynomial, then the degree d is one of its roots or $d \leq l_0$, or $d < \deg(G_0)$. Moreover, the existence of such l_0 will be proven for $\mathbb{K}$ being the field of real numbers. These results are based on the properties of the modules generated by special families of homogeneous symmetric polynomials.

A sufficient condition for the existence of a similar bound of the degree of a polynomial solution for an algebraic difference equation with G of arbitrary total degree and with variable coefficients will be proven as well.

Key words: algebraic difference equation, power-sum symmetric polynomial, partition, homogeneous symmetric polynomial

1. Introduction

This article addresses the problem of determining an upper bound of the degree d of a polynomial solution P of an algebraic difference equation (ADE for short) of the form

$$G\big(x\big)\big(P(x-\tau_1),\ldots,P(x-\tau_s)\big)+G_0(x)=0, \tag{1}$$

if such a solution $P \in \mathbb{K}[x]$ exists, where $G \in \mathbb{K}[x][x_1,\ldots,x_s]$ and $G_0 \in \mathbb{K}[x]$. Here $\mathbb{K}$ denotes a field of characteristic zero. $\mathbb{K}$ will be used as such throughout this article unless another meaning for $\mathbb{K}$ is stated explicitly. Also, we assume that $\tau_i \in \mathbb{K}$ are pairwise distinct.

Overview of the content of the presented work

It is known that, contrary to linear difference equations, there is no general theory for algebraic ones where G has total degree greater than 1. We study in detail the case of difference equations with constant coefficients

$$G\big(P(x-\tau_1),\ldots,P(x-\tau_s)\big)+G_0(x)=0, \tag{2}$$

where $G \in \mathbb{K}[x_1,\ldots,x_s]$.

This paper extends a previous article (Shkaravska and van Eekelen, 2014) of the same authors. The relevant notions and facts from that article will be recapitulated in *Section 2.* The new results are obtained by involving homogeneous symmetric polynomials. Necessary statements about these polynomials will be given in *Section 3.*

It will be shown in Theorem 4 in *Section 4* that given a difference equation (2) such that $G \in \mathbb{K}[x_1,\ldots,x_s]$ is quadratic, one can construct a countable family of univariate polynomials f_l with the following property: if $l_0 \geq 0$ is the minimal index such that f_{l_0} is a non-zero polynomial, then $f_{l_0}(d) = 0$ or $d \leq l_0$, or $d < \deg(G_0)$, where d is the degree of a polynomial solution P if such solution exists. The polynomial f_{l_0} is then an *indicial polynomial* for equation (2) similarly to an indicial polynomial defined for first-order linear difference systems in (Abramov and Barkatou, 1998). Moreover, in *Section 4* we will show that this result does not hold for polynomials G of degree three or greater due to a module-rank reason.

Note that the existence of a non-zero polynomial f_l in the family is not considered in Theorem 4. However, in Theorem 5 in *Section 5* we will prove the existence of a non-zero polynomial f_{l_0} for $\mathbb{K} = \mathbb{R}$ where $\mathbb{R}$ is the field of real numbers.

In *Section 6* we study difference equations of total degree $D \geq 2$ with polynomial coefficients. We will construct a family of 3 polynomials $f_0^*(u_0)$, $f_1^*(u_0)$, and $f_2^*(u_0)$ such that if one of them is non-zero, then an upper bound of the solutions' degrees is defined similarly to difference equations with constant coefficients. Furthermore, an example will be given of a quadratic equation with linear coefficients, such that it has a polynomial solution of any degree.

If the first-order theory of the field $\mathbb{K}$ is decidable, then knowing an upper bound of the degree of a possible polynomial solution for a given ADE makes it possible to find all of its polynomial solutions or to prove their absence. This is stated in Lemma 16 in *Section 7* and proven by means of *undetermined coefficients.*

Appendix *A* contains three subsections. First of all, tables with notations and definitions are given. This is followed by a bridge between previous results and the results in this paper. Second, full proofs are given of the auxiliary Lemmas 18 and 19. Finally, to support the reader in comprehending the statements for ADEs with variable coefficients, tables are given of the expressions used in these statements. We have used the computer algebra system Maxima to obtain these expressions. Further, the influence of G_0 on the existence of an upper bound of the degree of a polynomial solution is considered in detail. To support the reasoning, a table with necessary expressions is given and it is shown how they can be computed using Maxima.

Connection between the previous work of the authors with the work presented in this article

The work under consideration reassesses and extends the results of the earlier research (Shkaravska and van Eekelen, 2014). Let here and below $\mathbf{x}_D$ abbreviate a vector variable $(x_1, \ldots, x_D)$. Also one will use similar abbreviations: $\mathbf{u}_l$, $\mathbf{r}_d$ and $\mathbf{0}_l$ for $(u_1, \ldots, u_l)$, $(r_1, \ldots, r_d)$, and the 0-vector of length l respectively, where $r_1, \ldots, r_d$ denote the roots of a polynomial solution P. In the previous article one constructed a family of polynomials $S_l(u_0, \mathbf{u}_l)$ for equation (2), such that for $l \geq 0$ the polynomial $S_l\big(d, p_1(\mathbf{r}), \ldots, p_l(\mathbf{r})\big)$ is the coefficient of x^{Dd-l} on the left-hand side of equation (2) expanded as a symbolic polynomial in x. If $Dd-l > (D-1)d$ (that is $d > l$) and $(D-1)d \geq \deg(G_0)$ then clearly the identity $S_l\big(d, p_1(\mathbf{r}), \ldots, p_l(\mathbf{r})\big) = 0$ holds. The conditions under which $S_l(u_0, \mathbf{u}_l) = S_l(u_0, \mathbf{0}_l)$ for some $0 \leq l \leq 5$ as polynomials, were given. Therefore, under those conditions, $S_l(d, \mathbf{0}_l) = 0$ and $S_l(u_0, \mathbf{0}_l)$ could be taken as an indicial polynomial for equation (2). In general, if an indicial polynomial is not found among $S_0(u_0, \mathbf{0}_l), \ldots, S_5(u_0, \mathbf{0}_l)$ then the method does not give an answer.

In the current article it will be shown in Theorem 4 that for $D = 2$ the search for an indicial polynomial can be continued for all l until the first $S_l(u_0, \mathbf{0}_l)$ which is a non-zero polynomial is found. The improvement is done through the introduction of the $\mathbb{K}[u_0]$-modules generated by the power-sum products $p_1^{j_1}(\mathbf{x}_D) \cdots p_D^{j_D}(\mathbf{x}_D)$ with $j_1 + 2j_2 + \cdots + Dj_D = l$ being a partition of the number l, with every part at most D.

Related work

The case of quadratic difference equations with constant coefficients as considered in this article, is in a sense "dual" to the case considered in the article (Feng et al., 2008). In our article the polynomial G is of degree $D = 2$ and the number of the shifts τ_j is arbitrary whereas in (Feng et al., 2008) one considers equations of the form $G\big(P(x), P(x-\tau)\big) = G_0(x)$ where the polynomial G is of any degree D with 0 and τ as two shifts. In some kind of duality it is *degree two and any number of shifts versus any degree of G with two shifts*. In (Feng et al., 2008) it has been proven that if $G_0(x) \equiv 0$ and G is irreducible, then the degree of a polynomial solution is D.

In the book (Agarwal, 2000) one can find a detailed review of the known analytic and numerical methods for solving difference equations. In particular, in chapter 6 there are statements about the asymptotic and oscillating behavior of solutions of nonlinear equations. However, these results cannot be used for our purpose since the related statements in the quoted book either assume rather strict preconditions or are about lower bounds for non-oscillating solutions (whereas our aim is to bound the degree from above). For instance, in Section 6.17 one considers a non-linear equation of a very general form into which equation (2) fits in, but Theorem 6.17.1 about the asymptotic behavior of solutions assumes in its precondition that $G(x_1, \ldots, x_s) + G_0(n)$ is bounded by some linear w.r.t. x_i function of the form $\sum_{i=1}^{s} p_i(n) x_i$, which is not possible if G is a nonlinear polynomial in $x_1, \ldots, x_s$ and one looks for polynomial substitutions for x_s.

For an overview of related articles about analytical methods the reader is referred to (Shkaravska and van Eekelen, 2014). To our knowledge, after the publication of that article no new results appeared, that can be used to limit the degree of a polynomial solution. Researchers are mainly interested in wave-form solutions of algebraic difference equations, see, e.g., (Lee and Lee, 2016), whereas the research under consideration is devoted to polynomial solutions.

Speaking about algebraic methods for difference equations, one should mention the book (van der Put and Singer, 2003), devoted to Galois theory for linear difference equations. The present article might be a step towards developing a similar argument for non-linear equations.

Motivation and applications

Besides being mathematically intriguing objects, nonlinear ADE's have various applications. In particular, they appear in analyses of time consumption, memory consumption and other resource consumption of computer programs with recursive calls. For instance, for a natural number x, equations of the form $P(x) = G(x)\big(P(x-1), \ldots, P(x-s)\big)$ can represent the resource consumption in the recursive step x with $P(x-1), \ldots, P(x-s)$ representing the corresponding resource consumption on the previous steps. In general, resource consumption analysis often yields *inequalities* of the form $G(x)\big(P(x-1), \ldots, P(x-s)\big) \leq P(x)$. Studying inequalities is not the subject of this article. It is left to future research.

From the practical point of view, the results discussed in this article improve polynomial resource analysis of computer programs as, for instance, studied in Shkaravska et al. (2009). There the authors consider the size of output as a polynomial function on the sizes of inputs (Tamalet et al., 2008; Shkaravska et al., 2013). In the EU Charter project, the authors developed the ResAna tool (Shkaravska et al., 2007; van Kesteren et al., 2008; Shkaravska et al., 2010; Kersten et al., 2014) that applies polynomial interpolation to generate an upper bound on Java loop iterations. The tool requires from the user to input the degree of a possible solution. In (Shkaravska and van Eekelen, 2014) a partial result was proved that allowed in *some cases* to obtain the degree automatically.

Our results in Section 7 make it possible to derive automatically the degree of the polynomial in *all cases* for quadratic ADE's with constant coefficients and for a subclass of ADE's with variable coefficients.

2. Recapitulation: polynomial solutions of difference equations with constant coefficients

In (Shkaravska and van Eekelen, 2014) we established the existence of a *finite* family of 6 polynomials-candidates for an indicial polynomial for equation (2) where $D \geq 2$. If, for a given ADE, all the candidates from that family are zero polynomials, then the method proposed in that work does not give a bound for the ADE. In the present article we refine this result for quadratic equations showing that the family of the candidates in the quadratic case is countable, and the search can be continued until the first non-zero candidate is met.

To facilitate further reading, we recapitulate the machinery from (Shkaravska and van Eekelen, 2014) as far as it is necessary to prove the new results. Also, to illustrate notions and statements we will use the following difference equation as a *running example* in this article:

$$
\begin{aligned}
&P(x-1)P(x-1) - 3P(x-1)P(x-2) + \\
&\tfrac{5}{2}P(x-2)P(x-2) - \tfrac{1}{2}P(x-2)P(x-4) + \\
&\big(-P(x)\big) + 2P(x-1) - \tfrac{1}{8}P(x-2) = 0.
\end{aligned} \tag{3}
$$

Let $G_D(x_1, \dots, x_s) = \sum_{i_1+\ldots+i_s=D} a_{i_1\ldots i_s} x_1^{i_1} \cdots x_s^{i_s}$ denote the homogeneous part of degree D in the polynomial G. We introduce a reindexation φ for its coefficients in the following way.

Definition 1. *Reindexation φ is a map from the set of s-tuples $\{\mathbf{i} = (i_1, \dots, i_s) \mid \sum_{j=1}^{s} i_j = D\}$ to the set $\{\tau_1, \dots, \tau_s\}^D$ such that*

$$\varphi:\ (i_1, \dots, i_s) \mapsto (\underbrace{\tau_1, \dots, \tau_1}_{i_1}, \underbrace{\tau_2, \dots, \tau_2}_{i_2}, \dots, \underbrace{\tau_s, \dots, \tau_s}_{i_s}).$$

For instance, in equation (3) with $\tau_i = i$ for $i = 0, \dots, 4$ one has $G_2(x_0, x_1, x_2, x_3, x_4) = x_1^2 - 3x_1x_2 + \frac{5}{2}x_2^2 - \frac{1}{2}x_2x_4$, and G_2 can be considered as a polynomial in $x_1, \dots, x_4$. The reindexation φ is defined for the non-vanishing coefficients of G_2 in the following way:

	(i_1, i_2, i_3, i_4)	$\varphi(i_1, i_2, i_3, i_4)$
$x_1^2 = x_1x_1$	$(2, 0, 0, 0)$	$(1, 1)$
x_1x_2	$(1, 1, 0, 0)$	$(1, 2)$
$x_2^2 = x_2x_2$	$(0, 2, 0, 0)$	$(2, 2)$
x_2x_4	$(0, 1, 0, 1)$	$(2, 4)$

(4)

For the sake of convenience we introduce the notation for the image of the reindexation φ and the notations for the tuples of variables and values.

Notation 1. *The set T denotes the image $\varphi(\{\mathbf{i} = (i_1, \dots, i_s) \mid \sum_{j=1}^{s} i_j = D\})$.*

For instance, in the running example with $D = 2$, $s = 4$ and $\tau_i = i$, one has $T = \{(t_1, t_2) \mid 1 \le t_1 \le t_2 \le 4\}$. Clearly, the reindexation φ is a bijection from the set of all tuples $\{\mathbf{i} = (i_1, \dots, i_s) \mid \sum_{j=1}^{s} i_j = D\}$ to the set T since the shifts τ_i-s are pairwise distinct.

Notation 2. *Let $y_1, \dots, y_n$ be an arbitrary ordered collection of variables or values. Then $\mathbf{y}_n$ denotes the tuple $(y_1, \dots, y_n)$.*

In particular, the notations $\mathbf{t}_D$ and $\mathbf{r}_d$ abbreviate the tuple $(t_1, \dots, t_D) \in T$ and the tuple $(r_1, \dots, r_d)$ of the roots of P respectively.

We also rewrite the coefficients $a_{i_1\ldots i_s}$ by introducing $\alpha_{\mathbf{t}_D} := a_{i_1\ldots i_s}$ where $\mathbf{t}_D = \varphi(i_1,\ldots,i_s)$. For instance, for the running example $\alpha_{11} = a_{2000} = 1$, $\alpha_{12} = a_{1100} = -3$, $\alpha_{22} = a_{0200} = \frac{5}{2}$ and $\alpha_{24} = a_{0101} = -\frac{1}{2}$.

Let a polynomial P be represented via its roots: $P(x) = c(x-r_1)\cdots(x-r_d)$. The D-fold product $P(x-t_1)\cdots P(x-t_D)$ is equal to the product $c^D\prod_{j=1}^{D}\prod_{i=1}^{d}(x-r_i-t_j)$. For this product we are interested in the coefficients of the highest powers of x, namely x^{Dd-l}, where $0 \le l \le d-1$. This inequation appears due to the following reason: if a polynomial P of degree d solves equation (2) and $Dd-l > (D-1)d$ and $(D-1)d \ge \deg(G_0)$ for some $l \ge 0$ then the coefficients of x^{Dd-l} on the left-hand side of equation (2) must vanish. The inequation $l \le d-1$ is equivalent to $Dd-l > (D-1)d$.

More precisely, now we will study in detail the coefficient $\varepsilon_l(\mathbf{t}_D,\mathbf{r}_d)$ of x^{Dd-l} in the normalised product $\prod_{i=1}^{d}(x-r_i-t_j)$, where $0 \le l \le d-1$. The sums (t_j+r_i), where $1 \le j \le D$, $1 \le i \le d$ are the only roots of the polynomial $\prod_{j=1}^{D}\prod_{i=1}^{d}(x-r_i-t_j)$. Therefore, its coefficient $\varepsilon_l(\mathbf{t}_D,\mathbf{r}_d)$ is represented via *the elementary symmetric polynomials* $e_l(y_1,\ldots,y_m) := \sum_{1\le i_1<i_2<\cdots<i_l\le m} y_{i_1}\cdots y_{i_l}$ and $e_0(y_1,\ldots,y_m) := 1$ (Macdonald, 1979) in the standard way, with $m := Dd$:

$$\varepsilon_l(\mathbf{t}_D,\mathbf{r}_d) = (-1)^l e_l(t_1+r_1,\ldots,t_j+r_i,\ldots,t_D+r_d). \tag{5}$$

If the coefficients of x^{Dd-l} on the left-hand side of equation (2) must vanish then the roots $\mathbf{r}_d$ of $P(x)$ must satisfy the identity $\sum_{\mathbf{t}_D\in T}\alpha_{\mathbf{t}_D}c^D\varepsilon_l\big(\mathbf{t}_D,\mathbf{r}_d\big) = 0$ which is, due to $a_d \ne 0$, equivalent to the identity

$$\sum_{\mathbf{t}_D\in T}\alpha_{\mathbf{t}_D}\varepsilon_l\big(\mathbf{t}_D,\mathbf{r}_d\big) = 0. \tag{6}$$

Equation (6) does not give direct information about d since for any nonnegative integer index l the corresponding expression $\varepsilon_l(\mathbf{t}_D,\mathbf{r}_d)$ depends on d implicitly because d is the dimension of $\mathbf{r}_d$. To obtain an explicit equation for d from equation (6), we employ power-sum symmetric polynomials and the Newton-Girard formulæ (Macdonald, 1979):

$$e_l(y_1,\ldots,y_m) = (1/l)\sum_{\kappa=1}^{l}(-1)^{\kappa-1}e_{l-\kappa}(y_1,\ldots,y_m)p_\kappa(y_1,\ldots,y_m), \tag{7}$$

where the power-sum symmetric polynomial $p_\kappa(x_1,\ldots,x_m)$ of degree κ is

$$p_\kappa(x_1,\ldots,x_m) = x_1^\kappa + \cdots + x_m^\kappa \tag{8}$$

with $p_0(x_1,\ldots,x_m) = m$. As an instance we compute the values $p_\kappa(t_1,t_2)$ for $\kappa = 0,1,2$

and $1 \leq t_1 \leq t_2 \leq 4$ which will be used further when studying the running example (3):

	$(1,1)$	$(1,2)$	$(2,2)$	$(2,4)$
$p_0(t_1,t_2) = 2$	2	2	2	
$p_1(t_1,t_2) = t_1 + t_2$	2	3	4	6
$p_2(t_1,t_2) = t_1^2 + t_2^2$	2	5	8	20

Now, we note that by the definition of power-sum polynomials and the binomial formula one has

$$p_\kappa(\ldots, t_j + r_i, \ldots) = \sum_{j=1}^{D} \sum_{i=1}^{d} (t_j + r_i)^\kappa = \sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} p_\lambda(\mathbf{r}_d) p_{\kappa-\lambda}(\mathbf{t}_D). \tag{9}$$

Following Notation 2, we introduce *the shortcuts* $\mathbf{u}_l$ *and* $\mathbf{v}_l$ *which abbreviate the* l*-tuples of variables* $(u_1, \ldots, u_l)$ *and* $(v_1, \ldots, v_l)$ *respectively.* Substituting the tuple $(y_1, \ldots, y_m)$ by the tuple $(t_1 + r_1, \ldots, t_j + r_i, \ldots, t_D + r_d)$ and using equality (9) in the Newton-Girard formulæ (7), where $m = dD$, and recalling the connection between the polynomial roots and its coefficients via equation (5) one may see the idea behind the following construction.

Definition 2.

$$E_0\big(v_0, (), u_0, ()\big) := 1,$$
$$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) := -(1/l) \sum_{\kappa=1}^{l} E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa}) \left(\sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda} \right).$$

Let $\mathbf{p}_l(\mathbf{t}_D)$ *and* $\mathbf{p}_l(\mathbf{r}_d)$ *denote the* l*-tuples* $\big(p_1(\mathbf{t}_D), \ldots, p_l(\mathbf{t}_D)\big)$ *and* $\big(p_1(\mathbf{r}_d), \ldots, p_l(\mathbf{r}_d)\big)$ *respectively.* Using Definition 2 and identities (5) and (7), by induction on l one can prove that the following identity holds:

$$\varepsilon_l(\mathbf{t}_D, \mathbf{r}_d) = E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), d, \mathbf{p}_l(\mathbf{r}_d)\big). \tag{10}$$

This identity is proven as Lemma 2 in (Shkaravska and van Eekelen, 2014). As an instance for Definition 2, we consider the values of E_l for $l = 0, 1, 2$ [1]:

$$\begin{aligned} E_1() &= 1, \\ E_1(v_0, \mathbf{v}_1, u_0, \mathbf{u}_1) &= -v_1 u_0 - v_0 u_1, \\ E_2(v_0, \mathbf{v}_2, u_0, \mathbf{u}_2) &= -\tfrac{1}{2} v_2 u_0 + \tfrac{1}{2} v_1^2 u_0^2 - \tfrac{1}{2} v_0 u_2 - (v_1 - v_1 v_0 u_0) u_1 + \tfrac{1}{2} v_0^2 u_1^2. \end{aligned} \tag{11}$$

Identity (10) means that the value $E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), d, \mathbf{p}_l(\mathbf{r}_d)\big)$ is the coefficient of x^{Dd-l} in the normalised product $\frac{1}{c^D} P(x-t_1) \cdots P(x-t_D)$. For instance, $\varepsilon_1(\mathbf{t}_D, \mathbf{r}_d) = -d\, p_1(\mathbf{t}_D) - D\, p_1(\mathbf{r}_d)$ is the coefficient of x^{Dd-1} in this product.

[1] We have implemented the definition of E_l in Maxima. The corresponding script `ConstantCoefficients` is available on the Radboud Resource Analysis web-page `http://resourceanalysis.cs.ru.nl/#Algebraic%C2%A0Difference%C2%A0Equations`.

To describe the coefficients of x^{Dd-l} on the left-hand side of equation (2) after substituting a polynomial solution of degree d into it, we will need the following definition.

Definition 3. $S_l(u_0, \mathbf{u}_l) := \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{u}_l\big)$.

For $l = 0$, we will use the notation $S_0(u_0)$ since $u_0 = ()$ is empty. Using equation (6) and identity (10) one proves the next lemma.

Lemma 1. *If a polynomial P of degree d solves equation (2) with constant coefficients and $d > l$ for some $l \geq 0$ and $d \geq \deg(G_0)/(D-1)$ then $S_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big) = 0$.*

Proof. This statement is proven as Lemma 6 in (Shkaravska and van Eekelen, 2014). The conditions $d > l$ and $d \geq \deg(G_0)/(D-1)$ together imply that the coefficient $S_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big)$ of x^{Dd-l} on the left-hand side of equation (2) must vanish. □

Let l be a nonnegative integer. Applying Notation 2 we introduce shortcuts $\mathbf{i}_l$ *and* $\mathbf{j}_l$ *which denote the l-tuples $(i_1, \ldots, i_l)$ and $(j_1, \ldots, j_l)$ respectively, and $\mathbf{0}_l$ denotes the l-tuple $(0, \ldots, 0)$ of zeros.*

Definition 4. *A polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$, that is $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) = \sum_{\mathbf{i}_l} A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0) u_1^{i_1} \cdots u_l^{i_l}$.*

Note that despite $A_{\mathbf{0}_l}(v_0, \mathbf{v}_l)(u_0)$ is formally a polynomial of the variable v_0 as well, it can be proven that it does not depend on v_0 (see Lemma 18 in the Appendix). So, we will use the notation $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ instead. For instance, as one can see from the identities in (11), the values for $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ with $l = 0, 1, 2$ are

$$\begin{aligned} A_{()}(u_0) &= 1, \\ A_{(0)}(\mathbf{v}_1)(u_0) &= -v_1 u_0, \\ A_{(00)}(\mathbf{v}_2)(u_0) &= -\tfrac{1}{2} v_2 u_0 + \tfrac{1}{2} v_1^2 u_0^2. \end{aligned} \tag{12}$$

Applying Definition 4 it is easy to obtain the representation of $S_l(u_0, \mathbf{u}_l)$ as a polynomial in $\mathbf{u}_l$:

$$S_l(u_0, \mathbf{u}_l) = \sum_{\mathbf{i}_l} \Big(\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{(i_1, \ldots, i_l)}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) \Big) \cdot u_1^{i_1} \cdots u_l^{i_l}. \tag{13}$$

Now, we define polynomials that play a crucial role in the presented work.

Definition 5. *$B_{l,m}(\mathbf{v}_l)$ is the coefficient of u_0^m in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$.*

For instance, $B_{0,0}() = 1$, $B_{1,1}(v_1) = -v_1$ $B_{2,1}(\mathbf{v}_2) = -\frac{1}{2} v_2$ and $B_{2,2}(\mathbf{v}_2) = \frac{1}{2} v_1^2$. Moreover, for equation (3) we will need the values of $B_{l',m}(\mathbf{v}_{l'})$ at $\mathbf{p}_{l'}(t_1, t_2)$, where $l' = 0, 1, 2$:

$$
\begin{array}{l|rrrr}
 & (1,1) & (1,2) & (2,2) & (2,4) \\
\hline
B_{1,1} := -p_1(t_1,t_2) & -2 & -3 & -4 & -6 \\
B_{2,1} := -\frac{1}{2}p_2(t_1,t_2) & -1 & -\frac{5}{2} & -4 & -10 \\
B_{2,2} := \frac{1}{2}p_1^2(t_1,t_2) & 2 & \frac{9}{2} & 8 & 18
\end{array}
\tag{14}
$$

One can find more examples in (Shkaravska and van Eekelen, 2014). In that article it has been proven that for all $1 \leq l \leq 5$, for all $\mathbf{i}_l \neq \mathbf{0}_l$ there exist polynomials $H_{\mathbf{i}_l,l',m}(v_0, u_0)$ such that

$$
A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0) = \sum_{l'=0}^{l-1} \sum_{m=0}^{l'} H_{\mathbf{i}_l,l',m}(u_0, v_0) B_{l',m}(\mathbf{v}_{l'})
$$

that is the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is a $\mathbb{K}[u_0, v_0]$-linear combination of $B_{l',m}(\mathbf{v}_{l'})$ where $0 \leq m \leq l' \leq l-1$. This makes it possible to prove the following theorem, which is the main result of (Shkaravska and van Eekelen, 2014).

Theorem 1. *Let $P(x)$ be a polynomial solution of equation (2) and let d be its degree. If the set $\{l' \mid S_{l'}(u_0, \mathbf{0}_{l'})$ is a non-zero polynomial$\}$ is not empty and, moreover, $l := \min\{l' \mid S_{l'}(u_0, \mathbf{0}_{l'})$ is a non-zero polynomial$\} \leq 5$, then either $d \leq l$ or $d < \deg(G_0)/(D-1)\}$, or d must be one of the non-negative integer roots of $S_l(u_0, \mathbf{0}_l)$.*

For instance, for equation (3) one has that $S_0(u_0)$ and $S_1(u_0, 0)$ are equal to the zero polynomial and $S_2(u_0, \mathbf{0}_2) = \frac{1}{2}u_0(3 - u_0)$. Applying Theorem 1 one obtains that $l = 2$ and the degree of a polynomial solution is either $d = 0, 1, 2$ or it solves $S_2(u_0, \mathbf{0}_2) = 0$, that is $d = 3$. Indeed,

$$
\begin{aligned}
S_0(u_0) &= 1 - 3 + \tfrac{5}{2} - \tfrac{1}{2} = 0 \\
S_1(u_0, 0) &= 1 \cdot (-2) - 3 \cdot (-3) + \tfrac{5}{2} \cdot (-4) - \tfrac{1}{2} \cdot (-6) = 0 \\
S_2(u_0, 0, 0) &= \big(1 \cdot 2 - 3 \cdot \tfrac{9}{2} + \tfrac{5}{2} \cdot 8 - \tfrac{1}{2} \cdot 18\big)u_0^2 + \\
&\quad \big(1 \cdot (-1) - 3 \cdot (-\tfrac{5}{2}) + \tfrac{5}{2} \cdot (-4) - \tfrac{1}{2} \cdot (-10)\big)u_0 \\
&= -\tfrac{1}{2}u_0^2 + \tfrac{3}{2}u_0 = \tfrac{1}{2}u_0(3 - u_0).
\end{aligned}
\tag{15}
$$

The evidence of the fact that for $D = 2$ the theorem above can be refined appeared in the earlier research. The reader who is interested in a smooth transition from the old results to the new ones can find it in Subsection A.2 in the Appendix at the end of this text.

To refine Theorem 1 for quadratic ADE we will consider the modules generated by certain subfamilies of the polynomials $\{B_{l,m}\big(p_1(x_1, x_2), \dots, p_l(x_1, x_2)\big)\}_{m=0}^{l}$, which are, as we will show later, homogeneous and, obviously, symmetric in the variables x_1, x_2.

3. Homogeneous symmetric polynomials

As usual, $e_l(\mathbf{x}_n) = \sum_{1 \leq i_1 < \cdots < i_l \leq n} x_{i_1} \cdots x_{i_l}$ and $p_l(\mathbf{x}_n) = \sum_{i=1}^{n} x_i^l$ denote the elementary symmetric polynomial of degree l and the power-sum symmetric polynomial of degree l

respectively, with $e_0(\mathbf{x}_n) = 1$ and $p_0(\mathbf{x}_n) = n$. The following statement is known as the *fundamental theorem of symmetric polynomials* (van der Waerden et al., 2003).

Theorem 2. *Let $\mathbb{A}$ be a commutative ring with multiplicative identity $\mathbb{1}$. Then every symmetric polynomial $f(\mathbf{x}_n)$ from the subring of symmetric polynomials in $\mathbb{A}[\mathbf{x}_n]$ has a unique representation*

$$f(\mathbf{x}_n) = q\big(e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n)\big)$$

for some polynomial $q \in \mathbb{A}[\mathbf{x}_n]$.

Due to the Newton-Girard identities the elementary symmetric polynomial e_l is a rational linear combination of the products of the power-sum symmetric polynomials $p_1, \dots, p_l$. Therefore one can straightforwardly reformulate Theorem 2 in terms of power-sum symmetric polynomials:

Theorem 3. *Let $\mathbb{A}$ be a commutative ring containing the field $\mathbb{Q}$ of rational numbers (e.g. $\mathbb{A} = \mathbb{L}[x]$ where $\mathbb{L}$ is a field extension of $\mathbb{Q}$). Then every symmetric polynomial f from the subring of symmetric polynomials in $\mathbb{A}[\mathbf{x}_n]$ has a unique representation*

$$f(\mathbf{x}_n) = q\big(p_1(\mathbf{x}_n), \dots, p_n(\mathbf{x}_n)\big)$$

for some polynomial $q \in \mathbb{A}[\mathbf{x}_n]$.

Definition 6. *The weights $|\mathbf{i}_l|$ and $|\mathbf{j}_l|$ of the corresponding l-tuples are defined as the sums $i_1 + 2i_2 + \dots + li_l$ and $j_1 + 2j_2 + \dots + lj_l$ respectively.*

Notation 3. *The product $p_1^{j_1}(\mathbf{x}_n) \cdots p_n^{j_n}(\mathbf{x}_n)$ is denoted via $\pi^{\mathbf{j}_n}(\mathbf{x}_n)$.*

Notation 4. *Let $\mathbb{A}$ be a commutative ring containing $\mathbb{Q}$. Then $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_l|=l}$ denotes the $\mathbb{A}$-module generated by the products $\pi^{\mathbf{j}_n}$ such that $|\mathbf{j}_l| = l$.*

Lemma 2. *The set of all homogeneous symmetric polynomials of degree l from the ring $\mathbb{A}[\mathbf{x}_n]$ coincides with the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_l|=l}$.* [2]

Proof. First, it is easy to see that every polynomial from the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_l|=l}$ is homogeneous and symmetric since every generating polynomial $p_1^{j_1} \cdots p_n^{j_n}$ is symmetric and homogeneous of degree $j_1 + 2j_2 + \dots + nj_n = l$ w.r.t. $\mathbf{x}_n$. We use the fact that an $\mathbb{A}$-linear combination of homogeneous (resp. symmetric) polynomials is a homogeneous (resp. symmetric) polynomial of the same degree.

Second, the opposite inclusion holds as well, that is every homogeneous symmetric polynomial of degree l from $\mathbb{A}[\mathbf{x}_n]$ belongs to the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_l|=l}$. Indeed, it follows from Theorem 3 that for any symmetric polynomial $f(\mathbf{x}_n)$ of degree l there is a polynomial

[2] This statement mimics Corollary 7.7.2 from the book (Stanley, 1999). The difference is that Stanley does not consider symmetric polynomials but formal power series over infinite number of variables, i.e. constructions of the form $\sum_{\alpha_1+\alpha_2+\dots=n} x_1^{\alpha_1} x_2^{\alpha_2} \dots \in \mathbb{R}[x_1, x_2, \dots]$. For instance, $p_3(x_1, x_2, \dots, x_\ell)$ for some finite $\ell \geq 3$ is an element of the canonical generating set of the collection Λ^3 of such symmetric homogeneous functions since ℓ is not bounded. However $p_3(\mathbf{x}_2)$ does not belong to the canonical generating set of $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=3}$.

$q(y_1,\dots,y_n)$ such that $f = q(p_1,\dots,p_n)$ that is $f(\mathbf{x}_n) = \sum_{|\mathbf{j}_l|\le l} b_{\mathbf{j}_n} p_1^{j_1}(\mathbf{x}_n)\cdots p_n^{j_n}(\mathbf{x}_n)$ for some $b_{\mathbf{j}_n} \in \mathbb{A}$. For every $l_0 < l$ the subpolynomial $\sum_{|\mathbf{j}_l|=l_0} b_{\mathbf{j}_n} p_1^{j_1}(\mathbf{x}_n)\cdots p_n^{j_n}(\mathbf{x}_n)$ must vanish since f is homogeneous of degree l. Therefore $f(\mathbf{x}_n)$ is an $\mathbb{A}$-linear combination of the products $p_1^{j_1}\cdots p_n^{j_n}$ with $j_1+\cdots+nj_n = l$ and therefore it belongs to the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n}\rangle_{|\mathbf{j}_l|=l}$. □

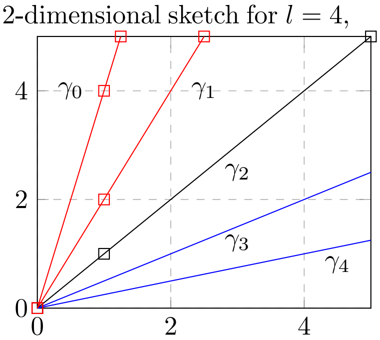
Lemma 3. *Let $f(x,y)$ be a symmetric homogeneous polynomial in $\mathbb{R}[x,y]$ of even degree l and let (x_m,y_m) be a collection of $l/2+1$ points, where $0 \le m \le l/2$, such that $y_m \ge x_m > 0$. Moreover, let all the lines $y = \frac{y_m}{x_m}x$ be pairwise distinct.*

If $f(x_m,y_m) = 0$ for all these nodes then $f(x,y)$ is the zero polynomial.

Proof. If $f(x_m,y_m) = 0$ then $f(\lambda x_m,\lambda y_m) = \lambda^l f(x_m,y_m) = 0$ due to the fact that f is homogeneous. The set $\gamma_m = \{(\lambda x_m,\lambda y_m) \mid \lambda \in \mathbb{R}\}$ is a parametric definition of the line defined by the points $(0,0)$ and (x_m,y_m).

Since f is symmetric, $f(x,y) = 0$ for all the points (x,y) that lie on the lines γ_{l-m}, where the line γ_{l-m} is symmetric to γ_m w.r.t. the line $y = x$. We also note that from the fact that the lines $y = y_m/x_m x$ are pairwise distinct it follows that at most one point lies on the line $y = x$. This implies that all together there are at least $l+1$ lines on which the polynomial $f(x,y)$ is equal to 0. Therefore, the polynomial $f(x,y)$ has at least $l+1$ zeros in the projective space $\mathbb{P}(\mathbb{R})$ of dimension one and therefore $f(x,y)$ is the zero polynomial. This concludes our proof. □

To illustrate the arguments in the proof above we give a 2-dimensional sketch for $l = 4$, with $(x_0,y_0) = (1,4)$, $(x_1,y_1) = (1,2)$ and $(x_2,y_2) = (1,1)$:



4. Properties of the A- and B-polynomials

In this section we consider the properties of the polynomials $A_{\mathbf{i}_l}(v_0,\mathbf{v}_l)(u_0)$ and $B_{l,m}(\mathbf{v}_l)$.

4.1. General properties of the A- and B-polynomials when D is an arbitrary integer greater than 1.

Notation 5. *As usual, $\mathbb{K}[u_0]$ stands for the ring of polynomials of the variable u_0 with coefficients in the field $\mathbb{K}$.*

Let $D \geq 1$. Following Notation 2, let $\mathbf{p}_l(\mathbf{x}_D)$ *denote the l-tuple of the power-sum symmetric polynomials* $\big(p_1(\mathbf{x}_D), \dots, p_l(\mathbf{x}_D)\big)$. Also let $\cong$ denote the natural ring isomorphism between the polynomial rings $\mathbb{K}[\mathbf{x}_D][u_0]$ and $\mathbb{K}[u_0][\mathbf{x}_D]$.

Lemma 4. *For any $l \geq 0$, and any $\mathbf{i}_l$ s. t. $0 \leq |\mathbf{i}_l| \leq l$, if the polynomial $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ from the ring $\mathbb{K}[\mathbf{x}_D][u_0] \cong \mathbb{K}[u_0][\mathbf{x}_D]$ is a non-zero polynomial then it is homogeneous in the variables $\mathbf{x}_D$ of degree $l - |\mathbf{i}_l|$, with coefficients in the ring $\mathbb{K}[u_0]$.*

Proof. We assign weights i and j to the variables u_i and v_j respectively. From the definition of E_l by induction on l one can easily show that all the terms of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ have the same weight l. From this it follows that given a tuple $\mathbf{i}_l$, the weight of the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0) \cdot u_1^{i_1} \cdot \dots \cdot u_l^{i_l}$ is l since it is a sum of terms of weight l. Therefore the weight of $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is $l - |\mathbf{i}_l|$. More precisely, each of its terms is a product of a monomial $v_1^{j_1} \cdots v_l^{j_l}$ with some polynomial from $\mathbb{K}[u_0, v_0]$ and has weight $l - |\mathbf{i}_l|$.

Now, we assign weight 1 to each variable x_j, and with this assignment the weight of $p_k(\mathbf{x}_D)$ is exactly its degree k. Since v_k and $p_k(\mathbf{x}_D)$ have the same weight k, an arbitrary term of $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ and the result of substituting the variables v_k by the polynomials $p_k(\mathbf{x}_D)$ in this term have the same weight $l - |\mathbf{i}_l|$. Obviously, since the degree of any term of $A_{\mathbf{i}}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ coincides with its weight and equal to $l - |\mathbf{i}_l|$, the degree of $A_{\mathbf{i}}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ is equal to $l - |\mathbf{i}_l|$ as well. Therefore it is a homogeneous polynomial in the variables $x_1, \dots, x_D$ over the ring $\mathbb{K}[u_0]$ as a sum of homogeneous terms of degree $l - |\mathbf{i}_l|$. $\square$

From Lemma 4, Lemma 2 and the fact that the polynomial $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ is symmetric in the variables $\mathbf{x}_D$ by its construction one immediately obtains the following result.

Lemma 5. *For any $l \geq 0$ and any $\mathbf{i}_l$, s. t. $0 \leq |\mathbf{i}_l| \leq l$, the polynomial $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ belongs to the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D| = l - |\mathbf{i}_l|}$.*

We note that if $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ is the zero polynomial then it trivially belongs to the module as the zero linear combination of its generators.

In Appendix A the following recurrent identity for $B_{l,m}$ is proven (Lemma 19):

$$B_{l,l-k}(\mathbf{v}_l) = -1/l \sum_{h=1}^{k+1} B_{l-h,l-k-1}(\mathbf{v}_{l-h}) v_h. \tag{16}$$

Using this identity with $v_\ell := p_\ell(\mathbf{x}_D)$, where $\ell \geq 0$, and applying the induction on $l \geq 0$ one immediately obtains the following statement.

Lemma 6. *For any $l \geq 0$ and $0 \leq m \leq l$, the polynomial $B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_D)\big) \in \mathbb{K}[\mathbf{x}_D]$ is a homogeneous polynomial in the variables $\mathbf{x}_D$ of degree l or it is the zero polynomial.*

We will see later that for $D = 2$ the family $\{B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_2)\big)\}_{m=1}^{l}$ contains an alternative generator set of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D| = l}$. However, for $D \geq 3$, this is not any more the case.

Lemma 7. *For $D \geq 3$ the family $\{B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_D)\big)\}_{m=1}^{l}$ does not contain a generator set of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D| = l}$.*

Proof. Since $\mathbb{K}[u_0]$ is a commutative ring one can apply rank reasons for a $\mathbb{K}[u_0]$-module as one would apply dimension reasons for a linear space over a field because for a commutative ring $\mathbb{A}$ an isomorphism $\mathbb{A}^m \cong \mathbb{A}^n$ implies $m = n$, see e.g. (Dummit and Foote, 2003), Exercise 2 of Section 10.3. This means that the size of any generator set of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D|=l}$ must be exactly the same as the size of its "canonical" generator set, which is the collection of the products $\pi^{\mathbf{j}_D}$ where $|\mathbf{j}_D| = l$.

For $l \geq 1$ the set $\{B_{l,m}(\mathbf{p}_l(\mathbf{x}_D))\}_{m=0}^{l}$ contains l non-zero polynomials (see Lemma 19), whereas the rank of $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D|=l}$ is the number $part_D(l)$ of the partitions $(1^{j_1}, 2^{j_2}, \dots, D^{j_D})$ of l such that $j_1 + 2j_2 + \cdots + Dj_D = l$. For $D = 2$ this number is $part_2(l) = \lfloor l/2 \rfloor + 1$. However, for $D = 3$ this is $part_3(l)$ which is the nearest integer number to $(l+3)^2/12$ (Stanley, 1997). It is a routine to check (e.g. by induction on $l \geq 6$, and direct calculations for $l = 0, \dots, 5$) that this number, which is an increasing function of l, may be less or equal to l only for $l \leq 5$, and otherwise the rank of $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D|=l}$ exceeds the number of non-zero polynomials in $B_{l,m}(\mathbf{p}_l(\mathbf{x}_D))$. In particular, for $D = 3$ and $l = 6$ one has that $(l+3)^2/12 = 81/12 = 6,75$ with the nearest integer number equal to 7.

In general, $part_D(l)$ is bounded from below by a polynomial of degree $D - 1$, see (Stanley, 1997) and a similar argument holds for any $D \geq 3$.[3] This concludes the proof of the Lemma. □

4.2. *The B-polynomials as module generators in the case of quadratic difference equations*

Everywhere in this subsection it is assumed that $D = 2$ and the power-sum polynomials are bivariate.

In Lemma 8 below we will show that for any $l \geq 1$ and any $0 \leq k \leq \lfloor l/2 \rfloor$ the polynomial $B_{l,l-k}(\mathbf{p}_l(\mathbf{x}_2))$ is a rational linear combination of the products $p_1^l, p_1^{l-2}p_2, p_1^{l-4}p_2^2, \dots, p_1^{l-2k}p_2^k$. Moreover, the coefficient of $p_1^{l-2k}p_2^k$ in this combination does not vanish. This will allow us to express for any $l \geq 0$ the generators $p_1^{l-2j}p_2^j$ of the module $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l}$ as linear combinations of the polynomials from the family $\{B_{l,l-k}(\mathbf{p}_l(\mathbf{x}_2))\}_{k=0}^{\lfloor l/2 \rfloor}$. This fact will be used to prove Lemma 10 which states that the family $\{B_{l,l-k}(\mathbf{p}_l(\mathbf{x}_2))\}_{k=0}^{\lfloor l/2 \rfloor}$ is a generator set of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l}$.

By Lemma 6 the polynomials $B_{l,l-k}(\mathbf{p}_l(\mathbf{x}_2))$ are homogeneous polynomials of degree l in the variables x_1 and x_2. Moreover, they are symmetric by construction. Therefore, they are linear combinations of the products $p_1^{l-2j}(\mathbf{x}_2)p_2^j(\mathbf{x}_2)$ by Lemma 2. However, to prove Lemma 8 we must know more about these linear combinations. For this we need the following auxiliary statement.

Lemma 8. *Given $D = 2$ and integer numbers l and k where $l \geq 0$ and $0 \leq k \leq \lfloor l/2 \rfloor$, for the corresponding polynomial $B_{l,l-k}(\mathbf{p}_l(\mathbf{x}_2))$ there exist rational numbers $b_{l,k,j}$ such that*

$$B_{l,l-k}(\mathbf{p}_l(\mathbf{x}_2)) = \sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(\mathbf{x}_2) p_2^j(\mathbf{x}_2), \tag{17}$$

[3] In item 10 under Corollary 1.4 of Stanley's book it is shown that the number of partitions $part'_k(n)$ of n into k parts is the same as the number of partitions $part_k(n)$ where the largest part is at most k. In Example 4.4.2 of the book it is shown that $part'_k(n)$ is a *quasipolynomial* of degree $k - 1$ with the minimal period equal to the least common multiple N of $1, \dots, k$, that is there are N polynomials f_i of degree at most $k - 1$ such that $part'_k(l) = f_i(l)$ once $l \equiv i \mod N$.

where the coefficient $b_{l,k,k}$ of $p_1^{l-2k}p_2^k$ does not vanish and has sign $(-1)^{l-k}$.

Proof. The proof is done by induction on l using identity (16). The base cases of induction are given by $l = 0, 1, 2$.

When $l = 0$ one has $B_{0,0} = 1$ and $k = 0$. The maximal power of $p_2(\mathbf{x}_2)$ in the representation of $B_{0,0}$ is obviously $0 = k$ and $b_{0,0,0} = 1$.

Now, let $l = 1$. Then for $k = 0$ one has $B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big) = B_{1,1}\big(\mathbf{p}_1(\mathbf{x}_2)\big) = -p_1(\mathbf{x}_2)$, see page 8. The maximal power of $p_2(\mathbf{x}_2)$ here is $0 = k$, and $b_{1,0,0} = -1$.

Further, for $l = 1$ the index $k = 1$ does not satisfy $k \le \lfloor \frac{l}{2} \rfloor$ so $B_{1,0} = 0$ is out of consideration.

Let $l = 2$. For $k = 0$ one has $B_{2,2}\big(\mathbf{p}_2(\mathbf{x}_2)\big) = \frac{1}{2}p_1^2(\mathbf{x}_2)$. The maximal power of $p_2(\mathbf{x}_2)$ here is $0 = k$, and $b_{2,0,0} = \frac{1}{2}$ has sign $(-1)^{2-0}$.

For $l = 2$ and $k = 1 = l/2$ one has $B_{2,1}\big(\mathbf{p}_2(\mathbf{x}_2)\big) = -\frac{1}{2}p_2(\mathbf{x}_2)$, see page 8. Trivially, $b_{2,1,1} = -\frac{1}{2}$ has sign $(-1)^{l-k}$.

Now, we continue with the induction step for $l \ge 3$. We prove the statement of the lemma separately for three possible cases concerning k: either $k = 0$, or $0 < k < l/2$, or $k = l/2$ for even l.

We start with the simplest case $k = 0$. In this case the sum on the right-hand side of identity (16) consists only of one summand $\frac{1}{l}B_{l-1,l-1}p_1$. This implies that $b_{l,0,0} = -\frac{1}{l}b_{l-1,0,0}$ and by the induction hypothesis for $l-1$ the coefficient $b_{l-1,0,0}$ has sign $(-1)^{l-1}$. Therefore the sign of $b_{l,0,0}$ is $(-1)^l$. Moreover, $B_{l,l}$ does not have occurrences of p_2^j with the power j greater than 0.

We continue with the case $0 < k < l/2$. To elaborate details, we fix $h \ge 1$ and consider the corresponding summand $B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(\mathbf{x}_2)\big)p_h(\mathbf{x}_2)$ from the right-hand side of identity (16). To be able to apply the induction hypothesis for $l-h$, we note that $B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(\mathbf{x}_2)\big) = B_{l-h,(l-h)-(k-h+1)}\big(\mathbf{p}_{l-h}(\mathbf{x}_2)\big)$ and, moreover, $0 < k < l/2$ implies that $k-h+1 \le \lfloor (l-h)/2 \rfloor$.[4] Applying the induction hypothesis we obtain that

$$B_{l-h,(l-h)-(k-h+1)}\big(\mathbf{p}_{l-h}(\mathbf{x}_2)\big) = \sum_{j=0}^{k-h+1} b_{l,k-h+1,j}\,p_1^{l-2j}(\mathbf{x}_2)p_2^j(\mathbf{x}_2)$$

where $b_{l-h,k-h+1,k-h+1}$ does not vanish and has the sign $(-1)^{l-h-(k-h+1)}$. This means that the maximal degree of p_2 in the expansion of $B_{l-h,(l-h)-(k-h+1)}$ is $k-h+1$. Further, the highest possible degree of p_2 in the expansion of p_h is obviously $\lfloor h/2 \rfloor$. Therefore, the product $p_1^{(l-h)-2(k-h+1)}p_2^{k-h+1}p_1^{h-2\lfloor h/2 \rfloor}p_2^{\lfloor h/2 \rfloor}$ yields the highest possible degree $g_{l,k}(h) := k-h+1+\lfloor h/2 \rfloor$ of p_2 in the expansion of the polynomial

$$B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(\mathbf{x}_2)\big)p_h(\mathbf{x}_2). \tag{18}$$

It is easy to check that $g_{l,k}(h)$ is a (non-strictly) decreasing function of $h \ge 1$. Therefore, its maximum is equal to k because it is achieved at $h = 1$ with $g_{l,k}(1) = k-1+1+0 = k$. Direct calculations show that this maximum is achieved at $h = 2$ as

---

[4] Indeed, if l is odd then $k < l/2$ implies $k \le (l-1)/2$. From this follows that $k-h+1 \le (l-1)/2 - h + 1 = (l-1-2h+2)/2 = (l+1-h-h)/2 \le (l-h)/2$ for $h \ge 1$. This implies that $k-h+1 \le \lfloor (l-h)/2 \rfloor$, since $k-h+1$ is integer. If l is even then $k < l/2$ implies $k \le (l-2)/2$ and therefore $k-h+1 \le (l-2)/2 - h + 1 = (l-2-2h+2)/2 = (l-h-h)/2 \le (l-h-1)/2$ for $h \ge 1$. If $l-h$ is odd then $(l-h-1)/2 = \lfloor (l-h)/2 \rfloor$ and therefore $k-h+1 \le \lfloor (l-h)/2 \rfloor$. If $l-h$ is even then $(l-h-1)/2 < (l-h)/2 = \lfloor (l-h)/2 \rfloor$ and therefore $k-h+1 < \lfloor (l-h)/2 \rfloor$.

well, with $g_{l,k}(2) = k-2+1+1 = k$. Checking at $h = 3$ yields $g_{l,k}(3) = k-3+1+1 = k-1$. From this follows that in order to compute $b_{l,k,k}$ by using identity (16), it is enough to consider the subsum with $h = 1, 2$ from the right-hand side of this identity, because only this subsum "contributes" to the highest degree k of p_2 in the expansion for $B_{l,l-k}$. This subsum is equal to $-\frac{1}{l}\big(B_{l-1,l-k-1}(\mathbf{p}_{l-1})p_1 + B_{l-2,l-k-1}(\mathbf{p}_{l-1})p_2\big)$. We note that $B_{l-1,l-k-1} = B_{l-1,(l-1)-k}$ and $B_{l-2,l-k-1} = B_{l-2,(l-2)-(k-1)}$ and therefore

$$b_{l,k,k} = -\frac{1}{l}(b_{l-1,k,k} + b_{l-2,k-1,k-1}). \tag{19}$$

By the induction hypothesis for $B_{l-1,(l-1)-k}$ one has $b_{l-1,k,k} \neq 0$ with its sign equal to $(-1)^{l-1-k}$, and by the induction hypothesis for $B_{l-2,(l-2)-(k-1)}$ one has $b_{l-2,k-1,k-1} \neq 0$ with its sign equal to $(-1)^{(l-2)-(k-1)}$. Therefore, both summands have the *same* sign $(-1)^{l-1-k}$ and $b_{l,k,k}$ has sign $(-1)^{l-k}$.

Now, let eventually $k = l/2$ for even l. Again, to elaborate details, we fix $h \geq 1$.

Firstly, let $h \geq 2$. One can show that $k-h+1 \leq \lfloor (l-h)/2 \rfloor$. Indeed, $k = l/2$ and $h \geq 2$ together imply that $k-h+1 = l/2-h+1 = (l-2h+2)/2 = (l+2-h-h)/2 \leq (l-h)/2$. Since $k-h+1$ is integer then $k-h+1 \leq \lfloor (l-h)/2 \rfloor$. Therefore we can repeat the fragment of the calculations for $0 < k < l/2$ to construct the function $g_{l,k}(h) = k - h + 1 - \lfloor h/2 \rfloor$, but this time it is defined on $h \geq 2$. The maximum value of this function equal to k and here it is achieved at $h = 2$.

Secondly, let $h = 1$. By Lemma 6 the polynomial $B_{l-1,l-k-1}\big(\mathbf{p}_2(\mathbf{x}_2)\big)$ is of degree $l - 1$ in x_1, x_2 and the maximal possible degree of the occurrences of $p_2(\mathbf{x}_2)$ in it is $\lfloor (l-1)/2 \rfloor$. Therefore, the maximal degree of the occurrences of p_2 in the product $B_{l-1,l-k-1}\big(\mathbf{p}_2(\mathbf{x}_2)\big)p_1(\mathbf{x}_2)$ is bounded from above by $\lfloor (l-1)/2 \rfloor$, which is less than k, since l is even and $k = l/2$.

From this it follows that for $k = l/2$ we have that $b_{l,k,k} = -\frac{1}{l} b_{l-2,k-1,k-1}$ does not vanish and has sign $(-1)^{1+\big(l-2-(k-1)\big)} = (-1)^{l-k}$.

This concludes the proof of the lemma.

□

Lemma 9. *Any polynomial of the form* $p_1^{l-2k}(\mathbf{x}_2)p_2^k(\mathbf{x}_2)$*, where* k *ranges from* 0 *to* $\lfloor l/2 \rfloor$*, is a* $\mathbb{Q}$*-linear combination of the polynomials*

$$B_{l,l}\big(\mathbf{p}_1(\mathbf{x}_2)\big), B_{l,l-1}\big(\mathbf{p}_l(\mathbf{x}_2)\big), \ldots, B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big).$$

Proof. Fix an arbitrary $l \geq 0$ and apply the induction on k using Lemma 8. For the base case $k = 0$ from equality (17) one trivially obtains $p_1^l(\mathbf{x}_2) = B_{l,l}\big(\mathbf{p}_l(\mathbf{x}_2)\big)/b_{l,0,0}$, where $b_{l,0,0} \neq 0$ has sign $(-1)^l$.

For the induction step we use again equality (17) of Lemma 8:

$$B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big) = b_{l,k,k}p_1^{l-2k}(\mathbf{x}_2)p_2^k(\mathbf{x}_2) + \sum_{j=0}^{k-1} b_{l,k,j}p_1^{l-2j}(\mathbf{x}_2)p_2^j(\mathbf{x}_2)$$

From this equality and $b_{l,k,k} \neq 0$ it follows that the product $p_1^{l-2k}p_2^k$ is a $\mathbb{Q}$-linear combination of the polynomials $B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)$ and $p_1^{l-2j}p_2^j$ where $j < k$. By the induction hypothesis each $p_1^{l-2j}p_2^j$ is a $\mathbb{Q}$-linear combination of polynomials

$$B_{l,l}\big(\mathbf{p}_l(\mathbf{x}_2)\big), B_{l,l-1}\big(\mathbf{p}_l(\mathbf{x}_2)\big), \ldots, B_{l,l-j}\big(\mathbf{p}_l(\mathbf{x}_2)\big).$$

The proof of the lemma follows immediately from this observation. □

Lemma 10. *The collection* $B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)$ *where* k *ranges from* 0 *to* $\lfloor l/2 \rfloor$ *is a generator set of the* $\mathbb{K}[u_0]$*-module* $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l}$.

Proof. The statement follows from Lemma 9 which shows that each canonical generator $p_1^{l-2k} p_2^k$, where $0 \le k \le \lfloor l/2 \rfloor$, for $D=2$ is a $\mathbb{Q}$-linear combination of $\{B_{l,l-j}\big(\mathbf{p}_l(\mathbf{x}_2)\big)\}_{j=0}^{k}$. □

Theorem 4. *Let* $D = 2$ *in equation (2) and let this equation have a polynomial solution of degree* d*. Let also* $l \ge 0$ *be such that* $S_l(u_0, \mathbf{0}_l)$ *for this equation is a non-zero polynomial and let the polynomials* $S_0(u_0), \ldots, S_{l-1}(u_0, \mathbf{0}_{l-1})$ *be all equal to the zero polynomial. Then* $S_l(u_0, \mathbf{u}_l) = S_l(u_0, \mathbf{0}_l)$*. Moreover,* $d \le l$*, or* $d < \deg(G_0)$ *or* $S_l(d, \mathbf{0}_l) = 0$*.*

Proof. Every polynomial $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ belongs to the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D|=l-|\mathbf{i}_l|}$, by Lemma 5. We set $k := |\mathbf{i}_l|$.

The fact that

$$S_{l-k}(u_0, \mathbf{0}_{l-k}) = \sum_{\mathbf{t}_2 \in T} \alpha_{\mathbf{t}_2} A_{\mathbf{0}_{l-k}}\big(2, \mathbf{p}_l(\mathbf{t}_2)\big)(u_0)$$

is the zero polynomial for $l-k<l$ means that each of its coefficients of u_0^m vanishes, that is

$$\sum_{\mathbf{t}_2 \in T} \alpha_{\mathbf{t}_2} B_{l-k,m}\big(\mathbf{p}_l(\mathbf{t}_2)\big) = 0 \tag{20}$$

for all $m = 0, \ldots, l-k$. Since the collection $\{B_{l-k,m}\big(\mathbf{p}_l(\mathbf{x}_2)\big)\}_{m=1}^{l-k}$ contains the generator set $\{B_{l-k,(l-k)-j}\big(\mathbf{p}_l(\mathbf{x}_2)\big)\}_{j=0}^{\lfloor (l-k)/2 \rfloor}$ of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l-k}$, the polynomial $A_{\mathbf{i}_l}\big(2, \mathbf{p}_l(\mathbf{x}_2)\big)(u_0)$ is a non-trivial $\mathbb{K}[u_0]$-linear combination of the polynomials $B_{l-k,m}\big(\mathbf{p}_{l-k}(\mathbf{x}_2)\big)$ where $k = |\mathbf{i}_l|$, and together with (20) this implies that for $k \ge 1$ the coefficient of $u_1^{i_1}, \ldots, u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$, which is equal to $\sum_{\mathbf{t}_2 \in T} \alpha_{\mathbf{t}_2} A_{\mathbf{i}_l}\big(2, \mathbf{p}_l(\mathbf{t}_2)\big)(u_0)$, vanishes. Indeed, since $A_{\mathbf{i}_l}\big(2, \mathbf{p}_l(\mathbf{x}_2)\big)(u_0) = \sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_2)\big)$ for some $c_{\mathbf{i}_l,m} \in K[u_0]$ then

$$\begin{aligned}
&\textstyle\sum_{\mathbf{t}_2 \in T} \alpha_{\mathbf{t}_2} A_{\mathbf{i}_l}\big(2, \mathbf{p}_l(\mathbf{t}_2)\big)(u_0) = \\
&\textstyle\sum_{\mathbf{t}_2 \in T} \alpha_{\mathbf{t}_2} \sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} B_{l-k,m}\big(\mathbf{p}_l(\mathbf{t}_2)\big) = \\
&\textstyle\sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} \sum_{(t_1,t_2) \in T} \alpha_{t_1 t_2} B_{l-k,m}\big(\mathbf{p}_l(\mathbf{t}_2)\big) = \\
&\textstyle\sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} 0 = 0.
\end{aligned}$$

Therefore, $S_l(u_0, \mathbf{u}_l) = S_l(u_0, \mathbf{0}_l)$.

Now, let $d > l$ and $d \ge \deg(G_0)$, which altogether implies that $2d - l > d \ge \deg(G_0)$. It means that the equality $S_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big) = 0$ must hold. Therefore $S_l(d, \mathbf{0}_l) = 0$ and $S_l(u_0, \mathbf{0}_l)$ is an indicial polynomial of the difference equation (2). This concludes the proof of the theorem. □

Note that to prove Theorem 4 we do not require that the field $\mathbb{K}$ is algebraically closed and/or ordered. However, when applying this theorem to instances of equation (2), it is convenient to consider the finite set of the shifts $\{\tau_1, \ldots, \tau_s\}$ as totally-ordered which is possible since any finite set can be totally-ordered.

5. Completing the procedure of bounding the degree of a solution for real quadratic difference equations

The existence of a non-zero polynomial in the family $\{f_l(u_0) := S_l(u_0, \mathbf{0}_l)\}_{l=0}^{\infty}$ is not considered in Theorem 4. In this section we will establish the existence of a non-zero polynomial in this family for $\mathbb{K} = \mathbb{R}$ where $\mathbb{R}$ is the field of real numbers. Therefore an upper bound of the degree d of a polynomial solution of equation (2) with $D = 2$ is defined and finite for $\mathbb{K} = \mathbb{R}$.

Everywhere in this section it is assumed that $\mathbb{K} = \mathbb{R}$, that is the polynomials G and G_0 have real coefficients and τ_i are real numbers. Moreover, as in the section above, it is assumed that $D = 2$. Without loss of generality one can assume that $\tau_1 < \cdots < \tau_s$ are positive. Otherwise one can consider a "shifted" difference equation:

$$G\big(P(x - \tau_1'), \ldots, P(x - \tau_s')\big) + G_0(x - \Delta) = 0, \tag{21}$$

where $\tau_i' = \tau_i + \Delta$ and Δ is some element of $\mathbb{R}$ such that all $\tau_i + \Delta > 0$. It is easy to see that the original difference equation has a polynomial solution P if and only if the shifted one has the same solution, using the fact that equation (2) holds for all x and therefore for all $x - \Delta$.

The following auxiliary lemma will be used in the proof of Lemma 13 below where we will see that if $S_0(u_0), \ldots, S_l(u_0, \mathbf{0}_l)$ are equal to the zero polynomial for a sufficiently large number $l \geq 0$ then the quadratic part of G vanishes.

Lemma 11. *Let $N > 0$ be an integer number and let $M = \{(t_1^{(m)}, t_2^{(m)})\}_{m=1}^{N}$ be a collection of positive real numbers such that all the ratios $t_2^{(m)}/t_1^{(m)} \geq 1$ are pairwise distinct. Then the $N \times N$ linear system*

$$\sum_{m=1}^{N} x_m B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(m)}, t_2^{(m)})\big) = 0, \tag{22}$$

where $0 \leq k \leq N-1$ ranges over the rows of the corresponding matrix, has only the trivial (i.e. all zero's) solution.

Proof. Let us assume the opposite, that is system (22) has a nontrivial solution, which we denote via $(x_1^0, \ldots, x_N^0)$. Then, the rows of the matrix are linearly dependent, that is for the vector-rows

$$\big(B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(1)}, t_2^{(1)})\big), \ldots, B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(N)}, t_2^{(N)})\big)\big), \tag{23}$$

where $0 \leq k \leq N-1$, there exists a nontrivial linear combination of them, equal to zero. This means that there exists a collection of $a_k \in \mathbb{R}$ such that

$$\sum_{k=0}^{N-1} a_k B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(m)}, t_2^{(m)})\big) = 0, \text{ for all } 1 \leq m \leq N, \tag{24}$$

where m ranges over the columns of the matrix. Consider the polynomial

$$F(\mathbf{x}_2) := \sum_{k=0}^{N-1} a_k B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(\mathbf{x}_2)\big).$$

It is homogeneous of degree $2N-2$ and symmetric in x_1, x_2 as a linear combination of homogeneous and symmetric polynomials. Moreover, it is given that it vanishes on the set $(t_1^{(m)}, t_2^{(m)})$ of N nodes, see equalities (24), such that all the corresponding N lines connecting the points $(0,0)$ and $(t_1^{(m)}, t_2^{(m)})$ are pairwise distinct. We apply Lemma 3 with $l = 2N-2$ and $l/2+1 = N$ for the polynomial $F(\mathbf{x}_2)$ to see that it vanishes everywhere. Therefore, there is a nontrivial linear combination of the polynomials $B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(\mathbf{x}_2)\big)$, where $k = 0, N-1$, such that it is equal to the zero polynomial, which contradicts the fact that the collection $\{B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(\mathbf{x}_2)\big)\}_{k=0}^{N-1}$ is a generator set for $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_D|\,=2N-2}$ where $D = 2$, see Lemma 10.

Therefore the assumption at the beginning of the proof is wrong, and the system has only the trivial solution.

□

Now we return to difference equation (2). Let R be the set of all the ratios t_2/t_1 where $\alpha_{(t_1,t_2)} \neq 0$. Let for each $r \in R$ the corresponding set M_r be defined as $M_r := \{(t_1,t_2) \mid \alpha_{(t_1,t_2)} \neq 0 \textit{ and } t_2/t_1 = r\}$. For instance, for the running example one has $R = \{1,2\}$ with $M_1 = \{(1,1),(2,2)\}$ and $M_2 = \{(1,2),(2,4)\}$.

We select from each set M_r a representative pair $\mathbf{t}_{2,r} := (t_{r1}, t_{r2})$. Trivially, for each pair $\mathbf{t}_2 = (t_1,t_2) \in M_r$ there is a number $\lambda_{r,\mathbf{t}_2} \in \mathbb{R}$ such that $\mathbf{t}_2 = \lambda_{r,\mathbf{t}_2}\mathbf{t}_{2,r}$. For the running example the natural representatives are $(1,1)$ and $(1,2)$ for $r=1$ and $r=2$ respectively, with $(2,2) = 2(1,1)$ and $(2,4) = 2(1,2)$.

Using these facts one can prove the following auxiliary statement.

Lemma 12. *For any $l \geq 0$ the coefficient of u_0^{l-k} in the polynomial $S_l(u_0, \mathbf{0}_l)$ is equal to*

$$\sum_{r\in R} \alpha'_{r,l} B_{l,l-k}\big(\mathbf{p}_l(\mathbf{t}_{2,r})\big),$$

where $\alpha'_{r,l} := \sum_{\mathbf{t}_2 \in M_r} \alpha_{\mathbf{t}_2} \lambda^l_{r,\mathbf{t}_2}$.

Proof. By the definitions of S_l, $A_{\mathbf{0}_l}$ and $B_{l,m}$, it follows that $S_l(u_0, \mathbf{0}_l)$ is equal to

$$\sum_{\mathbf{t}_2\in T} \alpha_{\mathbf{t}_2} A_{\mathbf{0}_l}\big(\mathbf{p}_l(\mathbf{t}_2)\big)(u_0) = \sum_{\mathbf{t}_2\in T} \alpha_{\mathbf{t}_2)} \sum_{k=0}^{l} B_{l,l-k}\big(\mathbf{p}_l(\mathbf{t}_2)\big) u^{l-k} =$$
$$\sum_{k=0}^{l} u_0^{l-k} \sum_{\mathbf{t}_2\in T} \alpha_{\mathbf{t}_2} B_{l,l-k}\big(\mathbf{p}_l(\mathbf{t}_2)\big) =$$
$$\sum_{k=0}^{l} u_0^{l-k} \sum_{r\in R} \sum_{\mathbf{t}_2\in M_r} \alpha_{\mathbf{t}_2} \sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(\mathbf{t}_2) p_2^j(\mathbf{t}_2)$$

where $b_{l,k,j} \in \mathbb{Q}$ are defined in Lemma 8. We fix some $r \in R$ and note that for a pair $\mathbf{t}_2 = \lambda_{r,\mathbf{t}_2}\mathbf{t}_{2,r} \in M_r$ and any non-negative integer ℓ by the definition of p_ℓ the equality $p_\ell(\mathbf{t}_2) = \lambda^\ell_{r,\mathbf{t}_2} t^\ell_{r,1} + \lambda^\ell_{r,\mathbf{t}_2} t^\ell_{r,2} = \lambda^\ell_{r,\mathbf{t}_2} p_\ell(\mathbf{t}_{2,r})$ holds. Therefore, for any $0 \leq j \leq k$ one has that

$$\sum_{\mathbf{t}_2\in M_r} \alpha_{\mathbf{t}_2} p_1^{l-2j}(\mathbf{t}_2) p_2^j(\mathbf{t}_2) = p_1^{l-2j}(\mathbf{t}_{2,r}) p_2^j(\mathbf{t}_{2,r}) \sum_{\mathbf{t}_2\in M_r} \alpha_{\mathbf{t}_2} \lambda^l_{r,\mathbf{t}_2}.$$

Indeed,

$$\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}p_1^{l-2j}(\mathbf{t}_2)p_2^j(\mathbf{t}_2)=$$
$$\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}(\lambda_{r,\mathbf{t}_2}t_{r1}+\lambda_{r,\mathbf{t}_2}^{l-2j}(\lambda_{r,\mathbf{t}_2}^2t_{r1}^2+\lambda_{r,\mathbf{t}_2}^2t_{r2}^2)^j=$$
$$\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}\lambda_{r,\mathbf{t}_2}^l(t_{r1}+t_{r2})^{l-2j}(t_{r1}^2+t_{r2}^2)^j=p_1^{l-2j}(\mathbf{t}_{2,r})p_2^j(\mathbf{t}_{2,r})\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}\lambda_{r,\mathbf{t}_2}^l.$$

Therefore, it is easy to check that if $\alpha'_{r,l}$ is defined as in the statement of the lemma then the coefficient of u^{l-k} in $S_l(u_0,\mathbf{0}_l)$ is equal to

$$\sum_{\mathbf{t}_2\in T}\alpha_{\mathbf{t}_2}B_{l,l-k}\big(\mathbf{p}_l(\mathbf{t}_2)\big)=\sum_{r\in R}\alpha'_{r,l}B_{l,l-k}\big(\mathbf{p}_l(\mathbf{t}_{2,r})\big),$$

as it is stated in the lemma. Indeed,

$$\sum_{(t_1,t_2)\in T}\alpha_{(t_1,t_2)}B_{l,l-k}\big(\mathbf{p}_l(t_1,t_2)\big)=$$
$$\sum_{(t_1,t_2)\in T}\alpha_{(t_1,t_2)}\big(\sum_{j=0}^{k}b_{l,k,j}p_1^{l-2j}(t_1,t_2)p_2^j(t_1,t_2)\big)=$$
$$\sum_{r\in R}\sum_{(t_1,t_2)\in M_r}\alpha_{(t_1,t_2)}\Big(\sum_{j=0}^{k}b_{l,k,j}p_1^{l-2j}(t_1,t_2)p_2^j(t_1,t_2)\Big)=$$
$$\sum_{r\in R}\Big(\sum_{j=0}^{k}b_{l,k,j}\sum_{(t_1,t_2)\in M_r}\alpha_{(t_1,t_2)}p_1^{l-2j}(t_1,t_2)p_2^j(t_1,t_2)\Big)=^{\text{by equation 5}}$$
$$\sum_{r\in R}\Big(\sum_{j=0}^{k}b_{l,k,j}\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}\lambda_{r,\mathbf{t}_2}^l p_1^{l-2j}(t_{r1},t_{r2})p_2^j(t_{r1},t_{r2})\Big)=$$
$$\sum_{r\in R}\Big(\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}\lambda_{r,\mathbf{t}_2}^l\sum_{j=0}^{k}b_{l,k,j}p_1^{l-2j}(t_{r1},t_{r2})p_2^j(t_{r1},t_{r2})\Big)=$$
$$\sum_{r\in R}\sum_{\mathbf{t}_2\in M_r}\alpha_{\mathbf{t}_2}\lambda_{r,\mathbf{t}_2}^l B_{l,l-k}\big(\mathbf{p}_l(t_{r1},t_{r2})\big)=$$
$$\sum_{r\in R}B_{l,l-k}\big(\mathbf{p}_l(t_{r1},t_{r2})\big)\alpha'_{(t_{r1},t_{r2})}.$$

□

Let $N:=|R|$ *be the cardinality of the set* R *for the equation* (2). We will prove now the following statement.

Lemma 13. *Let the sets* M_r *be singletons, except possibly* M_1*, containing all* $\mathbf{t}_2$ *for which* $t_1=t_2$*. Then* $S_{2N-2}(u_0,\mathbf{0}_l)$ *is a non-zero polynomial, or there exists* $0\le l\le s-1$ *such that* $S_l(u_0,\mathbf{0}_l)$ *is a non-zero polynomial.*

Proof. Firstly, we will show that if $S_{2N-2}(u_0,\mathbf{0}_{2N-2})$ is the zero polynomial then for any $t_1\ne t_2$ the corresponding coefficient $\alpha_{\mathbf{t}_2}$ vanishes and the non-zero coefficients of the quadratic part of G can be of the form $\alpha_{(t,t)}$ only. We fix $l_0=2N-2$ and assume that $S_{l_0}(u_0, \mathbf{0}_{l_0})$ is the zero polynomial, that is for all $0 \le k \le l_0$ its coefficient of u^{l_0-k} is zero. By Lemma 12, we obtain that for any $0 \le k \le N-1$ one has that $\sum_{r\in R} \alpha'_{r,l_0} B_{l_0,l_0-k}\big(\mathbf{p}_{l_0}(\mathbf{t}_{2,r})\big) = 0$. We consider a linear system $\sum_{r\in R} x_r B_{l_0,l_0-k}\big(\mathbf{p}_{l_0}(\mathbf{t}_{2,r})\big) = 0$ w.r.t. x_r, where $0 \le k \le N-1$ ranges over the rows of the matrix. By Lemma 11, setting $M = \{\mathbf{t}_{2,r} \mid r \in R\}$ there, one immediately obtains that the corresponding system has only the zero solution. Therefore, for all $r \in R$ one has $\alpha'_{r,l_0} = 0$. For any $r \ne 1$ the corresponding set M_r is a singleton of the form $M_r = \{\mathbf{t}_{2,r}\}$, and therefore $0 = \alpha'_{r,l_0} = \sum_{\mathbf{t}_2 \in M_r} \alpha_{\mathbf{t}_2} \lambda_{\mathbf{t}_2}^{l_0} = \alpha_{\mathbf{t}_{2,r}}$.

Secondly, we will show that the coefficients $\alpha_{(t,t)}$ vanish as well, if the polynomials $S_l(u_0, \mathbf{0}_l)$, where $0 \le l \le s-1$, are all equal to the zero polynomial. If for all $0 \le l \le s-1$ one has that $S_l(u_0, \mathbf{0}_l)$ is the zero polynomial then for any $0 \le l \le s-1$ the coefficient $\sum_{t\in\{\tau_1,\ldots,\tau_s\}} \alpha_{(t,t)} B_{l,l}\big(\mathbf{p}_l(t,t)\big)$ of u_0^l in the polynomial $S_l(u_0, \mathbf{0}_l)$ is zero. Since $B_{l,l}\big(\mathbf{p}_l(\mathbf{x}_2)\big) = b_{l,0,0} p_1^l(\mathbf{x}_2)$ where $b_{l,0,0} \ne 0$ by Lemma 8 for $k = 0$, this coefficient is equal to $b_{l,0,0} \sum_{t\in\{\tau_1,\ldots,\tau_s\}} \alpha_{(t,t)} p_1^l(t,t) = b_{l,0,0} \cdot 2 \sum_t \alpha_{(t,t)} t^l$. The determinant of the $s \times s$ system

$$\sum_{t\in\{\tau_1,\ldots,\tau_s\}} x_t t^l = 0, \text{ where } l = 0, 1, \ldots, s-1$$

is a non-zero Vandermonde determinant since all the shifts $\tau_1, \ldots, \tau_s$ are pairwise distinct. Therefore, this system only has the zero solution and all $\alpha_{(t,t)}$ vanish as well.

Therefore, the assumption that the polynomials $S_{2N-2}(u_0, \mathbf{0}_{2N-2})$ and $S_l(u_0, \mathbf{0}_l)$ for all $0 \le l \le s-1$ are the zero polynomials leads to vanishing of the quadratic part of the difference equation, which contradicts the fact that we consider quadratic equations. Therefore $S_{2N-2}(u_0, \mathbf{0}_{2N-2})$ is a non-zero polynomial or at least one of $S_l(u_0, \mathbf{0}_l)$, where $0 \le l \le s-1$, is a non-zero polynomial. The lemma is proven. □

To see that the condition of Lemma 13 does not influence the generality of the approach, one needs to consider a shifted equation of the form (21) with some properly chosen Δ. To provide the reader with an intuition we start with the running example of equation (3). In that equation the ratios $\frac{2}{1}$ and $\frac{4}{2}$ for two corresponding products $P(x-1)P(x-2)$ and $P(x-2)P(x-4)$ coincide. We note that one can construct a shifted equation which has the same polynomial solution as equation (3) and with such Δ that the ratios $\frac{2+\Delta}{1+\Delta}$ and $\frac{4+\Delta}{2+\Delta}$ are distinct. For instance, with $\Delta = 1$ one has $\frac{2+1}{1+1} = \frac{3}{2}$ and $\frac{4+1}{2+1} = \frac{5}{3}$. In general, the following lemma holds.

Lemma 14. *Given a finite set of pairs $U \subset \mathbb{R}^2$ such that it does not contain pairs of the form $(0,t)$ and pairs of the form (t,t), one can effectively define $\Delta \in \mathbb{R}^+$ such that for any two distinct pairs $(t_1,t_2) \ne (t'_1,t'_2) \in U$ one has $\frac{t_2+\Delta}{t_1+\Delta} \ne \frac{t'_2+\Delta}{t'_1+\Delta}$.*

Proof. For any two distinct elements $(t_1,t_2) \ne (t'_1,t'_2) \in U$ we will find $\Delta_{t_1,t_2,t'_1,t'_2}$ such that $\frac{t_2+\Delta_{t_1,t_2,t'_1,t'_2}}{t_1+\Delta_{t_1,t_2,t'_1,t'_2}} = \frac{t'_2+\Delta_{t_1,t_2,t'_1,t'_2}}{t'_1+\Delta_{t_1,t_2,t'_1,t'_2}}$ holds. Since this equation is equivalent to a polynomial equation w.r.t. $\Delta > 0$, there will be a finite number of the corresponding solutions $\Delta_{t_1,t_2,t'_1,t'_2}$ for all distinct pairs of pairs (t_1,t_2) and (t'_1,t'_2). Therefore, one can pick up an arbitrary $\Delta > 0$ which is distinct from all these $\Delta_{t_1,t_2,t_1',t_2'}$, and this Δ will satisfy the condition of the lemma.

We start with solving the equation $\frac{t_2+\Delta}{t_1+\Delta} = \frac{t_2'+\Delta}{t_1'+\Delta}$ w.r.t. $\Delta > 0$. This equation is equivalent to $(t_2+\Delta)(t_1'+\Delta) = (t_2'+\Delta)(t_1+\Delta)$ which is reduced to a linear one

$$(t_2 + t_1' - t_2' - t_1)\Delta = t_2' t_1 - t_2 t_1'.$$

For the sake of convenience we set $K := (t_2+t_1'-t_2'-t_1)$, $L := t_2't_1 - t_2t_1'$ and consider the equation $K\Delta = L$:

- $K \neq 0$; then the only solution is $\Delta_{t_1,t_2,t_1',t_2'} = L/K$;
- $K = 0, L \neq 0$, this case is impossible since $0 \cdot \Delta = L$ implies $L = 0$;
- $K = L = 0$; it is routine calculations to check that this case leads to a contradiction with the conditions of the lemma. Namely, $K = L = 0$ implies that the following system of equalities holds:

$$t_2 + t_1' - t_2' - t_1 = 0$$
$$t_2't_1 - t_2t_1' = 0$$

Since U does not contain pairs of the form $(0, t)$, we use $t_1' \neq 0$ and apply the substitution $t_2' = t_2t_1'/t_1$ (derived from the second equation) into the first equation. We obtain

$$t_2 - t_1 = t_2' - t_1' = t_2t_1'/t_1 - t_1' = t_1'(t_2/t_1 - 1) = t_1'(t_2 - t_1)/t_1$$

which implies that $t_2 = t_1$ or $t_1' = t_1$. The second option implies $t_2' = t_2$ via the first equation. Therefore, both options contradict the condition of the lemma.

Now, take any Δ which is distinct from all the $\Delta_{t_1,t_2,t_1',t_2'}$ where $(t_1, t_2) \neq (t_1', t_2') \in U$. This Δ makes $\frac{t_2+\Delta}{t_1+\Delta} \neq \frac{t_2'+\Delta}{t_1'+\Delta}$ for any $(t_1, t_2) \neq (t_1', t_2') \in U$. This concludes the proof of the lemma. □

The shifted equation for the running example, where $\Delta = 1$ has the form

$$\begin{aligned} &P(x-2)P(x-2) - 3P(x-2)P(x-3)+\\ &\tfrac{5}{2}P(x-3)P(x-3) - \tfrac{1}{2}P(x-3)P(x-5)+\\ &\big(-P(x-1)\big) + 2P(x-2) - \tfrac{1}{8}P(x-3) = 0 \end{aligned} \tag{25}$$

which does satisfy the condition of Lemma 13. It is a routine to show, e.g. using a computer algebra system, that the polynomials $S_0(u_0)$, $S_1(u_0, 0)$ and $S_2(u_0, \mathbf{0}_2)$ for the shifted equation (as well for the shifted equation for an arbitrary Δ) are exactly the same as for the original one, that is 0, 0 and $\frac{1}{2}u_0(3 - u_0)$ respectively.

Now, we are ready to prove the main result of the presented work.

Theorem 5. *If $\mathbb{K} = \mathbb{R}$ then for a difference equation of the form (2) with $D = 2$ there exists a countable family $\{f_l(u_0)\}_{l=0}^{\infty}$ of univariate polynomials and a number $0 \leq l_0 \leq \max\{s-1, 2N-2\}$ such that the polynomial $f_{l_0}(u_0)$ is non-zero. Moreover, if the difference equation has a polynomial solution of degree d then*

- *$d \leq l$,*
- *or $d < \deg(G_0)$,*

- *or d is a root of $f_{l_0}(u_0)$.*

Proof. Using Lemma 14 one can construct Δ such that the corresponding shifted difference equation satisfies the conditions of Lemma 13. A polynomial solution for the original difference equation is a solution for the shifted equation and vice versa. By Lemma 13 for the family $\{f_l(u_0) := S_l(u_0, \mathbf{0}_l)\}$ for the shifted equation there exists an index ($l \leq s-1$ or $l = 2N-2$) such that $f_l(u_0)$ does not vanish. We set l_0 to be the minimal such index l and apply Theorem 4 to complete the proof of the theorem. □

To give more details to the proof above, we take $U = T$ in Lemma 14 and obtain Δ such that the shifted difference equation satisfies the condition of Lemma 13, since all $\frac{t_2+\Delta}{t_1+\Delta}$ are pairwise distinct or $\frac{t_2+\Delta}{t_1+\Delta} = 1$. Therefore, that lemma can be applied to obtain the statement of this theorem.

Note that $\alpha^{\text{shifted}}_{(t_1+\Delta, t_2+\Delta)} = \alpha_{(t_1,t_2)}$ since $G\big(p(x-\tau_1'), \dots, p(x-\tau_s')\big)$ is obtained from $G\big(P(x-\tau_1), \dots, P(x-\tau_s)\big)$ via replacing every product $P(x-t_1)P(x-t_2)$ with the corresponding product $P\big(x-(t_1+\Delta)\big)P\big(x-(t_2+\Delta)\big)$ in the quadratic part of G and applying the corresponding substitutions in the linear part of G.

6. Algebraic difference equations of degree D with variable coefficients

In this section we study a difference equation (1) of total degree $D \geq 2$ with the polynomial coefficients where $G(x)(x_1, \dots, x_s) = \sum_{i_1+\cdots+i_s \leq D} a_{i_1\dots i_s}(x) x_1^{i_1} \cdots x_s^{i_s}$. Let H be the maximal degree of the coefficients $a_{i_1\dots i_s}(x)$ of the terms of degree D in the polynomial $G(x)(x_1, \dots, x_s)$ and let $P(x) = c(x-r_1)\cdots(x-r_d)$ be a hypothetical polynomial solution of the ADE. Again, we will construct polynomials $S_l^*(u_0, \mathbf{u}_l)$ such that $c^D S_l^*\big(d, \mathbf{p}_l(\mathbf{r}_d)\big)$ is the coefficient of x^{Dd+H-l} on the left-hand side of the given equation after substituting the symbol P in it by the hypothetical polynomial solution. We will show that if one of the polynomials $f_0^*(u_0) = S_0^*(u_0)$, $f_1^*(u_0) = S_1^*(u_0, 0)$ or $f_2^*(u_0) = S_2^*(u_0, 0, 0)$ is non-zero, then an upper bound of the degree d of P is defined similarly to difference equations with constant coefficients in Theorem 1, otherwise the method does not give an answer.

Contrary to quadratic difference equations with constant coefficients, degrees of polynomial solutions for difference equations with polynomial coefficients in general cannot be bounded because there are quadratic difference equations with polynomial coefficients that have a solution of any positive degree. Consider, for instance, the equation

$$P_n(x)P_n(x-1) - xP_n^2(x-1) + (x-1)P_n(x)P_n(x-2) = 0 \tag{26}$$

It is a routine to check that for an arbitrary positive integer number n the falling factorial $P_n(x) := x(x-1)\dots\big(x-(n-1)\big)$, which is a polynomial of degree n, solves this equation. Indeed, one has:

$$
\begin{aligned}
&P_n(x) - P_n(x-1) = \\
&x(x-1)\cdots\big(x-(n-1)\big) - (x-1)\cdots(x-n) = \\
&(x-1)\cdots\big(x-(n-1)\big)\big(x-(x-n)\big) = \\
&(x-1)\cdots\big(x-(n-1)\big)n = \\
&\tfrac{nP_n(x)}{x}.
\end{aligned}
$$

This implies that $n = \frac{x\left(P_n(x)-P_n(x-1)\right)}{P_n(x)}$ for all x, and therefore

$$
\begin{aligned}
&0 = n - n = \\
&\tfrac{x\left(P_n(x)-P_n(x-1)\right)}{P_n(x)} - \tfrac{(x-1)\left(P_n(x-1)-P_n(x-2)\right)}{P_n(x-1)} = \\
&\tfrac{xP_n(x-1)\left(P_n(x)-P_n(x-1)\right)-(x-1)P_n(x)\left(P_n(x-1)-P_n(x-2)\right)}{P_n(x)P_n(x-1)}
\end{aligned}
$$

which is equivalent to equation (26). Indeed

$$
\begin{aligned}
&0 = \\
&xP_n(x-1)\big(P_n(x) - P_n(x-1)\big) - (x-1)P_n(x)\big(P_n(x-1) - P_n(x-2)\big) = \\
&P_n(x)P_n(x-1) - xP_n^2(x-1) + (x-1)P_n(x)P_n(x-2)
\end{aligned}
$$

This proves that $P_n(x)$ solves that equation.

However, the earlier results for polynomial difference equations with constant coefficients of an arbitrary degree $D \geq 2$ in (Shkaravska and van Eekelen, 2014), to some extent still can be generalised for equations with polynomial coefficients.

The set of shifts $\{\tau_1, \ldots, \tau_s\} \subseteq \mathbb{K}$ is finite and therefore can be totally ordered. Let $\preccurlyeq$ denote a total order on this set. If $\mathbb{K} \subseteq \mathbb{R}$ then we assume that $\preccurlyeq$ is the usual order $\leq$ on real numbers.

Let m be a positive integer number and T_m denote the set of all non-decreasing m-tuples of the elements from the set $\{\tau_1, \ldots, \tau_s\}$. Formally, $T_m = \{(t_1, \ldots, t_m) \mid \tau_1 \preccurlyeq t_1 \preccurlyeq \cdots \preccurlyeq t_m \preccurlyeq \tau_s\}$. Let $\mathbf{t}$ range over the tuples from all the sets $T_1, \ldots, T_D$. Then equation (1) with polynomial coefficients has the following presentation:

$$
\sum_{m=1}^{D} \sum_{\mathbf{t} \in T_m} \alpha_{\mathbf{t}}(x) \cdot P(x-t_1)\cdots P(x-t_m) + G_0(x) = 0 \tag{27}
$$

where the coefficient of the m-fold product $P(x-t_1)\ldots P(x-t_m)$ is the polynomial $\alpha_{\mathbf{t}}(x) = w_{0,\mathbf{t}}x^{n_{\mathbf{t}}} + w_{1,\mathbf{t}}x^{n_{\mathbf{t}}-1} + \cdots + w_{n_{\mathbf{t}},\mathbf{t}}$ with the number $n_{\mathbf{t}}$ being the degree of the polynomial $\alpha_{\mathbf{t}}(x)$ and $w_{k,\mathbf{t}} \in \mathbb{K}$ being the coefficient of $x^{n_{\mathbf{t}}-k}$ in $\alpha_{\mathbf{t}}(x)$. For instance, for equation (26) one has $n_{(0,1)} = 0$, $n_{(1,1)} = 1$ and $n_{(0,2)} = 1$ with $\alpha_{(0,1)}(x) = 1, \alpha_{(1,1)}(x) = -x$ and $\alpha_{(0,2)}(x) = x - 1$ respectively.

As earlier, $\mathbf{t}_D$ abbreviates a (non-decreasing) D-tuple of the shifts. Also, let $\mathbf{w}_l$ denote the $(l+1)$-tuple of the variables $(w_0, \ldots, w_l)$, and $\mathbf{w}_{l,\mathbf{t}_D}$ denote the $(l+1)$-tuple of the values $(w_{0,\mathbf{t}_D}, \ldots, w_{l,\mathbf{t}_D}) \in \mathbb{K}^{l+1}$.

Let H_{D-1} denote the maximal degree of the polynomial coefficients of the $(D-1)$-fold products $P(x-t_1)\cdots P(x-t_{D-1})$ respectively. For instance, for equation (26) one has $H_{D-1}=0$ whereas $H=1$.

We consider now a product of the form $\alpha_{\mathbf{t}_D}(x)\cdot P(x-t_1)\cdots P(x-t_D)$. If the degree of $\alpha_{\mathbf{t}_D}$ is some $n<H$, we assign $w_{k,\mathbf{t}_D}=0$, where $k+n<H$. To compute the coefficients of x^{Dd+H-l} in this product, where $0\le l\le Dd+H$, one will need the following definition, based on the rule of multiplication of two polynomials applied to the polynomial $\alpha_{\mathbf{t}_D}(x)$ and the symbolic polynomial $P(x-t_1)\cdots P(x-t_D)$:

Definition 7. $E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) := \sum_{k=0}^{l} E_k\big(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l\big) w_{l-k}.$

Using the rule of the multiplication of two polynomials, it is easy to prove the following lemma.

Lemma 15. *Let $0\le l\le Dd+H$. The coefficient of x^{Dd+H-l} in the product $\alpha_{\mathbf{t}_D}(x)\cdot P(x-t_1)\cdots P(x-t_D)$ is equal to $c^D E_l^*\big(\mathbf{w}_{l,\mathbf{t}_D}, D, \mathbf{p}_l(\mathbf{t}_D), d, \mathbf{p}_l(\mathbf{r}_d)\big)$.*

Proof. Fix some integer numbers k_1 and k_2 such that $0\le k_1\le H$ and $0\le k_2\le Dd$. Since $w_{H-k_1,\mathbf{t}_D}$ is the coefficient of $x^{H-(H-k_1)}=x^{k_1}$ in $\alpha_{\mathbf{t}_D}(x)$ and the value $E_{Dd-k_2}\big(D, \mathbf{p}_{Dd-k_2}(\mathbf{t}_D), d, \mathbf{p}_{Dd-k_2}(\mathbf{r}_d)\big)$ is the coefficient of $x^{Dd-(Dd-k_2)}=x^{k_2}$ in the symbolic product $P(x-t_1)\cdots P(x-t_D)$, divided by c^D, by the polynomial-multiplication rule one has that the coefficient of x^{Dd+H-l} in the normalised product $\frac{1}{c^D}\alpha_{\mathbf{t}_D}(x)\cdot P(x-t_1)\cdots P(x-t_D)$, is equal to:

$$\sum_{k_1+k_2=Dd+H-l} w_{N-k_1,\mathbf{t}_D} E_{Dd-k_2}\big(D, \mathbf{p}_{Dd-k_2}(\mathbf{t}_D), d, \mathbf{p}_{Dd-k_2}(\mathbf{r}_d)\big) =^{l_1:=H-k_1, l_2:=Dd-k_2}$$
$$\sum_{Dd+H-l_1-l_2=Dd+H-l} w_{l_1,\mathbf{t}_D} E_{l_2}\big(D, \mathbf{p}_{l_2}(\mathbf{t}_D), d, \mathbf{p}_{l_2}(\mathbf{r}_d)\big) =$$
$$\sum_{l_1+l_2=l} w_{l_1,\mathbf{t}_D} E_{l_2}\big(D, \mathbf{p}_{l_2}(\mathbf{t}_D), d, \mathbf{p}_{l_2}(\mathbf{r}_d)\big),$$

where $0\le l_1\le H$ and $0\le l_2\le Dd$. The conclusion of the lemma follows by setting $l_1=l-k$ and $l_2:=k$ in the equality above. □

Now, we introduce the following definition.

Definition 8. $S_l^*(u_0, \mathbf{u}_l) := \sum_{\mathbf{t}_D\in T} E_l^*(\mathbf{w}_{l,\mathbf{t}_D}, D, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{u}_l).$

Obviously $c^D S_l^*\big(d, \mathbf{p}_l(\mathbf{r}_d)\big)$ is the coefficient of x^{Dd+H-l} on the left-hand side of equation (1) if $Dd+H-l>(D-1)d+H_{D-1}$ and $(D-1)d+H_{D-1}\ge \deg(G_0)$. Therefore, it must vanish when these inequations hold. Using a computer algebra system it is easy to prove the following statement.

Theorem 6. *Let an algebraic difference equation (1) be given. If there exists an integer number $0\le l\le 2$ such that $f_l^*(u_0):=S_l^*(u_0,\mathbf{0}_l)$ is a non-zero polynomial and for any $0\le l'\le l-1$ the corresponding $f_{l'}:=S_{l'}^*(u_0,\mathbf{0}_{l'})$ is the zero polynomial, then the following holds for the degree d of a polynomial solution of equation (1):*

- $d \leq l - H + H_{D-1}$,
- *or* $d < \big(\deg(G_0) - H_{D-1}\big)/(D-1)$,
- *or* d *is a root of* $f_l^*(u_0)$.

Proof. We fix l that satisfies the condition of the lemma. Assuming that the first two alternative conclusions of the lemma do not hold, we will show that then the third one must follow.

If $d > l - H + H_{D-1}$ and $d \geq (\deg(G_0) - H_{D-1})/(D-1)$ then $S_l^*\big(d, \mathbf{p}_l(\mathbf{r}_d)\big) = 0$ and the schema of the proof is the same as the schema of the proof of Theorem 1 for the polynomials with constant coefficients. We consider the expression $E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ as a polynomial in $\mathbb{K}[\mathbf{w}_l, v_0, \mathbf{v}_l][u_0][\mathbf{u}_l]$ and define the polynomial $A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$ as its coefficient of $u_1^{i_1} \cdots u_l^{i_l}$. Consequently, the polynomial $B^*_{l,m}(\mathbf{w}_l, \mathbf{v}_l)$ is the coefficient of u_0^m in the polynomial $A^*_{\mathbf{0}_l}(\mathbf{w}_l, \mathbf{v}_l)(u_0)$.

Using symbolic computations it is easy to check that for $l = 0, 1, 2$ and $\mathbf{i}_l \neq \mathbf{0}_l$ the polynomial $A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$ is a $\mathbb{K}[u_0, v_0]$-linear combination of the polynomials $B^*_{l,m}(\mathbf{w}_l, \mathbf{v}_l)$, where $0 \leq m \leq l - 1$. In Subsection A.5 in the Appendix one can find the tables which contain the expressions for the polynomials A^*. Moreover, the symbolic coefficients for the corresponding $\mathbb{K}[u_0, v_0]$-linear combinations are given there as well.

For $l = 3$ the expression for $A^*_{100}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$ is not a $\mathbb{K}[u_0, v_0]$-linear combination of the polynomials $B^*_{l,m}(\mathbf{w}_l, \mathbf{v}_l)$, where $0 \leq l \leq 2$ and $0 \leq m \leq l - 1$. This can be shown by solving the linear system w.r.t. the unknown coefficients of the hypothetical $\mathbb{K}[u_0, v_0]$-linear combination for A^*_{100}. The system is derived by equating the corresponding coefficients of the monomials $w_0^{k_0} w_1^{k_1} w_2^{k_2} v_1^{j_1} v_2^{j_2}$ in A^*_{100} and in the linear combination. The system is inconsistent and therefore the linear combination does not exist. [5]

From this it follows that if $S_0^*(u_0) = \sum_{\mathbf{t}_D \in T} w_{0,\mathbf{t}_D} = 0$ then the dependency on u_1 vanishes in $S_1^*(u_0, u_1) = \sum_{\mathbf{t}_D \in T} E_1^*(\mathbf{w}_{l,\mathbf{t}_D}, D, \mathbf{p}_l(\mathbf{t}_D), u_0, u_1)$. If $S_1^*(u_0, 0)$ is a non-zero polynomial then it is an indicial polynomial for the difference equation under consideration. Otherwise, the dependencies on u_1 and u_2 vanish in $S_2^*(u_0, u_1, u_2)$. If $S_2^*(u_0, 0, 0)$ is a non-zero polynomial then it is an indicial polynomial. Otherwise the method does not give an answer. In this case the coefficient of u_1 in $S_3^*(u_0, \mathbf{u}_3)$ is equal to $-1/2 \sum_{\mathbf{t}_D \in T} p_2(\mathbf{t}_D) w_{0,\mathbf{t}_D}$ and does not necessarily vanish in general. Therefore $S_3^*(u_0, \mathbf{u}_3)$ is not reducible to a 1-variable polynomial of u_0 and cannot be taken as an indicial polynomial. □

Note that the coefficients $w_k(y_1, \ldots, y_D)$ considered as functions given by their values $w_k(\mathbf{t}_D) = w_{k,\mathbf{t}_D}$ can be viewed as the polynomials obtained by the D-variate interpolation in the nodes $(\mathbf{t}_D, w_{k,\mathbf{t}_D})$, where $\mathbf{t}_D \in T$. In principle, the polynomials $w_k(y_1, \ldots, y_D)$ can be made symmetric by adding the nodes of the form $\big(\sigma(\mathbf{t}_D), w_{k,\mathbf{t}_D}\big)$ where σ runs over all the permutations of the variables $y_1, \ldots, y_D$. However such polynomials are not necessarily homogeneous. Still there may be possibilities to refine Theorem 6 using homogeneous symmetric polynomials. Studying such possibilities will be a subject of our future work.

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[5] The calculations are implemented in the Maxima script `VariableCoefficients` which can be found on the Radboud Resource Analysis web-page.

7. Constructing polynomial solutions given an upper bound of their degrees

In this section we will show that if the first-order theory for the field $\mathbb{K}$ is decidable then knowing an upper bound of the degree of a possible polynomial solution for a given ADE allows to find all its polynomial solutions or to establish their absence. A remarkable example of fields with decidable first-order theories are real closed fields.

We continue with the running example given by equation (3). As it has been shown in Section 2, for this ADE the degree d of its possible polynomial solution can be found among the numbers $0, 1, 2, 3$. Introducing a symbolic polynomial $P_{a_3,a_2,a_1,a_0}(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and substituting it into equation (3), one obtains an algebraic system w.r.t. the parameters a_3, a_2, a_1, a_0 by equating to zero the coefficients of $x^6, \dots, x^1, x^0$ on the l.h.s. of equation (3) after this substitution. Using a computer algebra system one can solve this system. The coefficients of $x^6 = x^{2d-0}$, $x^5 = x^{2d-1}$ and $x^4 = x^{2d-2}$, where $d = 3$, vanish since they are equal to $S_0(3) = 0$, $S_1(3, 0)$ and $S_2(3, \mathbf{0}_2) = 0$ respectively. The symbolic coefficients of $x^3, x^2, x, 1$ are equal to

$$\begin{aligned}
C_3(a_0, a_1, a_2, a_3) &= (168a_3^2 + 7a_3)/8 \\
C_2(a_0, a_1, a_2, a_3) &= -(888a_3^2 + (-168a_2 + 24a_1 + 42)a_3 - 8a_2^2 - 7a_2)/8 \\
C_1(a_0, a_1, a_2, a_3) &= (1584a_3^2 + (-592a_2 + 168a_1 - 72a_0 + 36)a_3)/8+ \\
&\quad ((8a_1 - 28)a_2 + 7a_1)/8 \\
C_0(a_0, a_1, a_2, a_3) &= -(952a_3^2 + (-528a_2 + 224a_1 - 168a_0 + 8)a_3)/8+ \\
&\quad (24a_2^2 + (24a_0 - 12)a_2 - 8a_1^2 + 14a_1 - 7a_0)/8
\end{aligned} \tag{28}$$

respectively.[6] Solving the system $C_3(a_0, a_1, a_2, a_3) = 0, \dots, C_0(a_0, a_1, a_2, a_3) = 0$ w.r.t. $a_3, \dots, a_0$, yields an infinite number of solutions amongst of which there are complex ones and the trivial one $a_3 = \dots = a_0 = 0$. Real and rational tuples solving this system exist as well. For instance, there is a subfamily of solutions defined by the relations $a_3 = -1/24, a_1 = -(192a_2^2 + 5)/24, a_0 = (256a_2^3 + 20a_2 - 5)/12$, where a_2 is free.

In general, the following statement holds.

Lemma 16. *If the first-order theory of the field $\mathbb{K}$ is decidable then for any ADE of the form (1) there exists a finite deterministic algorithm which for an arbitrary nonnegative integer d answers the question if this ADE has a polynomial solution of maximal total degree d or not.*

Proof. Given an ADE and an arbitrary integer $d \geq 0$, the decision procedure for $\mathbb{K}$ takes as an input the finite system of algebraic equations w.r.t. the parameters $a_d, \dots, a_1, a_0$ induced by equating to zero the coefficients of x^l on the l.h.s. of the ADE, which is instantiated with the parametric polynomial $P_{a_d,\dots,a_1,a_0}(x) := a_dx^d + \dots + a_1x + a_0$. The procedure decides if the system is solvable or not. Moreover, if the procedure finds $a_d, \dots, a_1, a_0$ constructively, then the corresponding polynomials $P_{a_d,\dots,a_1,a_0}(x)$ are solutions of the ADE, by their construction. □

[6] One can find the corresponding Maxima script `RunningExample` on the above mentioned Radboud Resource Analysis web-page.

Generally speaking, real closed fields are decidable because they admit quantifier elimination, which is currently implemented in different versions of *cylindrical algebraic decomposition* (Collins, 1998). However, from the practical point of view these procedures are not always efficient if one needs to find a polynomial solution for an ADE or to prove its absence. Since the search for such a solution, given an upper bound for its degree, amounts to solving a finite system $\mathbb{S}$ of polynomial equations w.r.t solution's coefficients, one can use methods involving Gröbner bases. For instance, one can test if the system $\mathbb{S}$ has solutions applying Hilbert's Weak Nullstellensatz (Cox et al., 2015) in the following way. One computes a minimal Gröbner basis G of $\mathbb{S}$ and checks if 1 is an element in the set G. If yes, there is no solutions for the system $\mathbb{S}$ and therefore the ADE does not have a polynomial solution. Otherwise, there is at least one tuple $(a_d, \ldots, a_0) \in \mathbb{K}^{d+1}$ on which all the equations in $\mathbb{S}$ vanish simultaneously, and this tuple defines a polynomial solution of the ADE. Gröbner-basis methods can be also used to compute the tuples $(a_d, \ldots, a_0)$ explicitly. For instance, if there are only finitely many such tuples one can utilize the Shape Lemma (Winkler, 1996) to find them.

If the first-order theory of the field (or, more generally, ring) $\mathbb{K}$ is not decidable then in general it is not decidable if a given ADE in $\mathbb{K}$ has a polynomial solution in $\mathbb{K}[x]$ of degree at most d. It can be proven by establishing a connection between Diophantine equations and algebraic difference equations. How it is done in general is shown in subsection A.6 Appendix. Here, we consider a simple example which gives an idea behind the connection between ADEs and Diophantine equations. We will construct an ADE which has a polynomial solution $P(x) = a_1x + a_0$ of degree $d = 1$ in $\mathbb{Q}[x]$ if and only if the corresponding equation $a_0^D + a_1^D = 1$ has rational solutions $(a_0, a_1) \in \mathbb{Q}^2$. It is known that for $D \geq 3$ this equation does not have solutions w.r.t. (a_0, a_1), other than $(0, 1)$ and $(1, 0)$. This is the version of Fermat's last theorem for rational numbers. For the equation $a_0^D + a_1^D = 1$ the corresponding ADE is derived in the following way. First, one considers the parametric system w.r.t. a_0 and a_1:

$$\begin{aligned} a_1x + a_0 &= P(x) \\ a_1(x-1) + a_0 &= P(x-1) \end{aligned} \tag{29}$$

The corresponding determinants from Cramer's rule for this system are $\Delta(x) = x - (x-1) = 1$, $\Delta_1(x, P) = P(x) - P(x-1)$, and $\Delta_0(x, P) = xP(x-1) - (x-1)P(x)$. The ADE that corresponds to the Diophantine equation $a_0^D + a_1^D = 1$ is obtained via the substitutions $a_0 := \Delta_0(x, P)/\Delta(x)$ and $a_1 := \Delta_1(x, P)/\Delta(x)$ into this Diophantine equation:

$$\big(xP(x-1) - (x-1)P(x)\big)^D + \big(P(x) - P(x-1)\big)^D = 1. \tag{30}$$

This example illustrates the complexity of the problem of solving ADE's in $\mathbb{Q}[x]$ even when an upper bound of the degree of a possible polynomial solution is given. Recall that the existence of an algorithm which, given a Diophantine equation, decides if it has a rational solution or not (Hilbert's 10th problem for rational numbers), is still an open problem.

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A. Appendix

A.1. Notations and definitions

This section contains a table of notations and a table of definitions used in this article.

Notation	Meaning	Page
$\mathbb{K}$	a field of characteristic zero	1
$\mathbf{t}_D, \mathbf{r}_d$	$(t_1, \ldots, t_D), (r_1, \ldots, r_d)$	5
$\mathbf{u}_l, \mathbf{v}_l$	$(u_1, \ldots, u_l), (v_1, \ldots, v_l)$	7
$\mathbf{p}_l(\mathbf{t}_D), p_l(\mathbf{r}_d))$	the tuples of the values	
	of the power-sum polynomials	
	$\big(p_1(\mathbf{t}_D), \ldots, p_l(\mathbf{t}_D)\big)$,	
	$\big(p_1(\mathbf{r}_d), \ldots, p_l(\mathbf{r}_d)\big)$	
	respectively	7
$\mathbf{i}_l, \mathbf{j}_l, \mathbf{0}_l$	$(i_1, \ldots, i_l), (j_1, \ldots, j_l), (0, \ldots, 0)$	8
$\lvert\mathbf{i}_l\rvert$	$i_1 + 2i_2 + \cdots + li_l,$ the weight of $\mathbf{i}_l$	6
$\mathbb{Q}$	the field of rational numbers	10
$\mathbf{x}_n$	the tuple of the variables $(x_1, \ldots, x_n)$	5
$\pi^{\mathbf{j}_n}$	the product of the power-sum polynomials	
	$p_1^{j_1} \cdots p_n^{j_n}$	3
$\mathbb{R}$	the field of real numbers	2
$\mathbf{w}_l$	the tuple of the variables $(w_0, \ldots, w_l)$	23
$\mathbf{w}_{l,\mathbf{t}_D}$	the tuple of the values $(w_{0,\mathbf{t}_D}, \ldots, w_{l,\mathbf{t}_D})$	23

(A.1)

Definition	Brief description	Page
φ	is a map from the set of s-tuples	
	of nonnegative integer numbers such that	
	$\varphi: \ (i_1, \ldots, i_s) \mapsto$	
	$\big(\underbrace{\tau_1, \ldots, \tau_1}_{i_1}, \underbrace{\tau_2, \ldots, \tau_2}_{i_2}, \ldots, \underbrace{\tau_s, \ldots, \tau_s}_{i_s}\big)$	
	where $i_1 + \cdots + i_s = D$	5
T	the image $\varphi(\{\mathbf{i} = (i_1, \ldots, i_s) \mid \sum_j i_j = D\})$	5
$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$	$-(1/l)\left(\sum_{\kappa=1}^{l} E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa}) \cdot \right.$	7
	$\left.\Big(\sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda}\Big)\right)$	

$S_l(u_0, \mathbf{u}_l)$	$\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{u}_l\big)$	8
$A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$	is the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$	8
	in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$.	
	For $\mathbf{i}_l = \mathbf{0}_l$ this polynomial	
	does not depend on v_0,	
	therefore one can write $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$	
$B_{l,m}(\mathbf{v}_l)$	is the coefficient of u_0^m in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$	8
$E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$	$\sum_{k=0}^{l} E_k\big(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l\big) w_{l-k}$	24
$A_{\mathbf{i}_l}^*(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$	is the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$	24, the proof
	in $E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$	of Theorem 6
$B_{l,m}^*(\mathbf{w}_l, \mathbf{v}_l)$	is the coefficient of u_0^m in $A_{\mathbf{0}_l}^*(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$	ibid.

A.2. Bridge from the old result to the present ones

If the condition of Theorem 1 does not hold, that is, for all $0 \le l \le 5$ the polynomials $S_l(u_0, \mathbf{0}_l)$ are equal to the zero polynomial, then, in general, $S_6(u_0, \mathbf{u}_6)$ and $S_6(u_0, \mathbf{0}_6)$ do not have to be equal as polynomials and therefore $S_6(u_0, \mathbf{0}_6)$ cannot be taken as an indicial polynomial. More precisely, the following statement holds (Shkaravska and van Eekelen, 2014).

Lemma 17. If for any $0 \le l \le 5$ the polynomial $S_l(u_0, \mathbf{0}_l)$ is the zero polynomial, then

$$S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6) + (1/8)(u_1^2 - u_2 u_0) \sum_{\mathbf{t}_D \in T} p_2^2(\mathbf{t}_D)\alpha_{\mathbf{t}_D}. \tag{A.2}$$

As one can see, $S_6(u_0, \mathbf{u}_6)$ depends not only on the variable u_0 which represents the degree d but on the difference $u_1 u_0 - u_2^2$ with variables u_1 and u_2 representing the power-sums $p_1(\mathbf{r}_d)$ and $p_2(\mathbf{r}_d)$ of the unknown roots of a solution. However, it was proven in Corollary 2 of (Shkaravska and van Eekelen, 2014) that for $D = 2$ the coefficient $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2)$ of $u_1 u_0 - u_2^2$ vanishes if $S_0(u_0), \ldots, S_5(u_0, \mathbf{0}_5)$ are all equal to the zero polynomial. We reconsider the proof from that article to provide the reader with an intuition behind the arguments used in the work under consideration. Moreover, we will see why the same result does not hold for $D > 2$.

We observe that the condition $S_4(u_0, \mathbf{0}_4) \equiv 0$ induces the system of equations

$$\begin{cases} 1/24 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_1^4(t_1, t_2) = 0, \text{ the coefficient of } u_0^4 \\ -1/4 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2(t_1, t_2) p_1^2(t_1, t_2) = 0, \text{ the coefficient of } u_0^3 \\ 1/3 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_3(t_1, t_2) p_1(t_1, t_2) + 1/8 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2) = 0, \\ \qquad\qquad \text{the coefficient of } u_0^2 \\ -1/4 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_4(t_1, t_2) = 0, \text{ the coefficient of } u_0^1. \end{cases} \tag{A.3}$$

Note that for $D = 2$ the product p_3p_1 is equal to $-1/2p_1^4 + 3/2p_2p_1^2$. This can be shown by direct calculations, using the definition of $p_l(t_1, t_2) = t_1^l + t_2^l$. Indeed $p_1^3 = p_3 + 3t_1^2t_2 + 3t_1t_2^2 = p_3 + 3t_1t_2(t_1 + t_2)$. Now, use $t_1t_2 = 1/2(p_1^2 - p_2)$ to obtain $p_3 = p_1^3 - 3/2p_1(p_1^2 - p_2) = -1/2p_1^3 + 3/2p_1p_2$. This implies that $B_{4,2}(p_1, p_2) = 1/3p_3p_1 + 1/8p_2^2 = -1/6p_1^4 + 1/2p_1^2p_2 + 1/8p_2^2$.

From this equality and the equations for the coefficients of u_0^4, u_0^3 and u_0^2 in the system above it follows that $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2) = 0$. Therefore the term containing u_1 and u_2 in S_6 vanishes and $S_6(u_0, \mathbf{0}_6)$ is an indicial polynomial, unless it is the zero polynomial.

For $D = 3$ equating coefficients of $u_0^4, \ldots, u_0^1$ in $S_4(u_0, \mathbf{0}_4)$ to zero does not imply that $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2) = 0$. To see this, again consider the system (A.3). Note that $p_4 = 1/6p_1^4 + 1/2p_2^2 + 4/3p_1p_3 - p_1^2p_2$. This can be checked by direct calculations, e.g. using a computer algebra one can check that $1/6(x + y + z)^4 + 1/2(x^2 + y^2 + z^2)^2 + 4/3(x + y + z)(x^3 + y^3 + z^3) - (x + y + z)^2(x^2 + y^2 + z^2) = x^4 + y^4 + z^4$. Therefore, the last equation can be discarded because it is an linear combination of the first three ones, where the first and the third equations are multiplied by -1, and the second equation is multiplied by 4. Therefore one obtains a system of 3 equations with 4 variables:

$$\begin{cases} 1/24X = 0 \\ -1/4Y = 0 \\ 1/3U + 1/8V = 0, \end{cases}$$

where X stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_1^4(t_1, t_2)$, Y stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2(t_1, t_2) p_1^2(t_1, t_2)$, U stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_3(t_1, t_2) p_1(t_1, t_2)$ and V is for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2)$. It is obvious that $X = Y = 0$, but then either U or V is a free variable and $V = 0$ cannot be established.

Now, consider $D \geq 4$. Again, equating coefficients of $u_0^4, \ldots, u_0^1$ in $S_4(u_0, \mathbf{0}_4)$ to zero, in general does not imply that $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2) = 0$. Indeed, in this case one obtains the system of 4 equations with 5 variables, where X, Y, U, V are defined as above and Z stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_4(t_1, t_2)$, with no auxiliary equations between these variables.

A.3. Auxiliary lemmas

In this section we prove two auxiliary lemmas. Lemma 18 allows to omit the variable v_0 in the lists of the variables of the polynomials $A_{\mathbf{0}_l}$, where $l \geq 0$. Moreover, it is used in the proof of Lemma 19. Lemma 19 is used in the proofs of Lemma 6 and Lemma 8.

Lemma 18. *For the polynomials $A_{\mathbf{0}_l}$ the following inductive identity holds:*

$$\begin{aligned} &A_{()}()(u_0) = 1, \\ &A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0) = -(1/l) \sum_{h=1}^{l} A_{\mathbf{0}_{l-h}}(\mathbf{v}_{l-h})(u_0) \cdot u_0 v_h. \end{aligned} \tag{A.4}$$

This also means that for any $l \leq 0$ the polynomial $A_{\mathbf{0}_l}$ does not depend on the variable v_0. Moreover, for $l \geq 1$ there are no non-zero u_0-free terms in $A_{\mathbf{0}_l}$.

Proof. We recall Definition 2 of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$:

$$E_0\big(v_0, (), u_0, ()\big) := 1,$$

$$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) := -(1/l)\sum_{h=1}^{l} E_{l-h}(v_0, \mathbf{v}_{l-h}, u_0, \mathbf{u}_{l-h})\left(\sum_{\lambda=0}^{h}\binom{h}{\lambda}u_\lambda v_{h-\lambda}\right).$$

To obtain the non-zero $\mathbf{u}_l$-free terms in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ one needs to set the indices λ in the products of the form $\binom{h}{\lambda}u_\lambda v_{h-\lambda}$ only to 0, since these terms are constituted by the summands that contain only the products of the form $u_0 v_h$ where $h \geq 1$. Otherwise, assume that there is a $\mathbf{u}_l$-free subterm Ku_0^m of E_l which is the result of polynomial-ring axiomatics applied to some sum $\sum_{i_0 i_1 \ldots i_l} K_{i_0 i_1 \ldots i_l} u_1^{i_1} \cdots u_l^{i_l} u_0^{i_0}$ in E_l. This is impossible in the ring of polynomials over fields of characteristic zero, unless both, the sub-term and the sum. are equal to zero.Then, identity (A.4) follows immediately. The proven recursive identity is turned in a program in the computer algebra system so that the symbolic value $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ can be obtained for any fixed $l \geq 0$. By induction on l it follows that $A_{\mathbf{0}_l}$ does not contain occurrences of v_0. Moreover, from the proven identity it follows that for $l \geq 1$ there are no non-zero u_0-free terms in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ because all the summands on the right-hand side of this identity are divisible by u_0. □

Lemma 19. Let $l \geq 1$ and $0 < k < l$. Then the identity

$$B_{l,l-k}(\mathbf{v}_l) = -1/l\sum_{h=1}^{k+1} B_{l-h,l-k-1}(\mathbf{v}_{l-h})v_h$$

holds and $B_{l,0}(\mathbf{v}_l) = 0$.

Proof. We use Lemma 18 above. First, the fact that for $l \geq 1$ there are no non-zero u_0-free terms in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ implies that $B_{l,0}(\mathbf{v}_l) = 0$ since it is the u_0-free term of $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ by its definition.

Second, identity (A.4) implies that for the coefficient $B_{l,m}(\mathbf{v}_l)$ of u_0^m in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$, where $1 \leq m < l$, the following recurrent identity holds:

$$\begin{aligned} B_{l,m}(\mathbf{v}_l) &= -1/l\sum_{h=1}^{l} B_{l-h,m-1}(\mathbf{v}_{l-h})v_h \\ &= -1/l\sum_{l-h\geq m-1, h=1}^{l} B_{l-h,m-1}(\mathbf{v}_{l-h})v_h \\ &= -1/l\sum_{h=1}^{l-m+1} B_{l-h,m-1}(\mathbf{v}_{l-h})v_h. \end{aligned}$$

We introduce the index k by assigning $k := l - m$. Then, the identity above implies the statement of the lemma: $B_{l,l-k}(\mathbf{v}_l) = -1/l\sum_{h=1}^{k+1} B_{l-h,l-k-1}(\mathbf{v}_{l-h})v_h$. This concludes the proof of the lemma. □

A.4. Other properties of E- and A-polynomials

In this section we consider a number of properties of the polynomials $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ and $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$. These properties are used when one considers the influence of the polynomial $G_0(x)$ on the existence of an upper bound of the degree of the solutions of a given ADE. They may be used in future research as well.

Lemma 20. Let $S_0(u_0)$ be the zero polynomial and $|\mathbf{i}_l| = l$. Then for all $l \geq 1$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l$. Recall that for any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 0$ which means that for any term that occurs in $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = \cdots = j_l = 0$. From this it follows that $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0) = A_{\mathbf{i}_l}(v_0)(u_0)$ since it does not contain terms with occurrences of v_k, where $k \geq 1$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}(D)(u_0) = A_{\mathbf{i}_l}(D)(u_0) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} = 0$ due to $S_0(u_0) = \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} = 0$. □

Lemma 21. Let $S_1(u_0, 0)$ be equal to the zero polynomial and $|\mathbf{i}_l| = l - 1$. Then for all $l \geq 2$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l-1$. Recall that for any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 1$ which means that for all the terms that occur in the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = 1$ and $j_2 = \cdots = j_l = 0$. This implies that $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is of the form $K(u_0, v_0)v_1$ for some $K \in \mathbb{K}[u_0, v_0]$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) = K(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1(\mathbf{t}_D) = 0$ due to the fact that $S_1(u_0, 0) = u_0 \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1(\mathbf{t}_D)$ is the zero polynomial. □

Lemma 22. Let $S_2(u_0, \mathbf{0}_2)$ be equal to the zero polynomial and $|\mathbf{i}_l| = l - 2$. Then for all $l \geq 3$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l - 2$. Recall that any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 2$. From this it follows that for all the terms that occur in the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = 2$ and $j_2 = \cdots = j_l = 0$, or $j_2 = 1$ and $j_1 = j_3 = \cdots = j_l = 0$. This implies that $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is of the form $K_1(u_0, v_0)v_1^2 + K_2(u_0, v_0)v_2$ for some $K_1, K_2 \in \mathbb{K}[u_0, v_0]$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) =$ $K_1(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1^2(\mathbf{t}_D) + K_2(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_2(\mathbf{t}_D) = 0$ due to the fact that $S_2(u_0, 0, 0)$ is the zero polynomial, because its coefficients of u_0 and u_0^2 are proportional to the sums $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_2(\mathbf{t}_D)$ and $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1^2(\mathbf{t}_D)$ respectively.
□

Lemma 23. Let $S_3(u_0, \mathbf{0}_3)$ be equal to the zero polynomial and $|\mathbf{i}_l| = l - 3$. Then for all $l \geq 4$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l-3$. Recall that for any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 3$. From this it follows that for all the terms that occur in the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = 3$ and $j_2 = \cdots = j_l = 0$, or $j_1 = j_2 = 1$ and $j_3 = \cdots = j_l = 0$, or $j_3 = 1$ and $j_1 = j_2 = j_4 = \cdots = j_l = 0$. This implies that $A_{\mathbf{i}_l}(u_0)(v_0, \mathbf{v}_l)$ is of the form $K_1(u_0, v_0)v_3^2 + K_2(u_0, v_0)v_1v_2 + K_3(u_0, v_0)v_3$ for some $K_1, K_2, K_3 \in \mathbb{K}[u_0, v_0]$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to

$$\begin{aligned}\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) = {} & K_1(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1^3(\mathbf{t}_D) + \\ & K_2(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1(\mathbf{t}_D) p_2(\mathbf{t}_D) + \\ & K_3(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_3(\mathbf{t}_D) \\ = {} & 0\end{aligned}$$

due to the fact that $S_3(u_0, \mathbf{0}_3)$ is the zero polynomial. □

Lemma 24. Let $S_0(u_0), \ldots, S_3(u_0, \mathbf{0}_3)$ be all equal to the zero polynomial and let $l \geq 5$. Then if some term of $S_l(u_0, \mathbf{u}_l)$ contains u_{l-4} then u_{l-4} occurs in this term only linearly. In other words, there are no terms in $S_l(u_0, \mathbf{u}_l)$ which contain the products of u_{l-4} and any other $u_\lambda^{i_\lambda}$ with $\lambda \geq 1$ and $i_\lambda > 0$, and, in particular, there are no terms with powers of u_{l-4} which are higher than 1.

Proof. The statement follows from Lemmata 20, 21, 22, 23. Indeed, if such a term had occurred in $S_l(u_0, \mathbf{u}_l)$ then due to $l - 4 \geq 1$ this term would have satisfied the inequation $i_1 + 2i_2 + \cdots + (l-4)i_{l-4} + \cdots + li_l \geq (l-4) + 1 = l - 3$. Then either $|\mathbf{i}_l| = l$, or $|\mathbf{i}_l| = l-1$, or $|\mathbf{i}_l| = l-2$, or $|\mathbf{i}_l| = l-3$, and one can apply one of the lemmata listed above. Therefore the term vanishes in $S_l(u_0, \mathbf{u}_l)$. □

Lemma 25. Let $l \geq 5$. Then the coefficient $A_{(\mathbf{0}_{l-5}, 1, \mathbf{0}_4)}(v_0, \mathbf{v}_l)(u_0)$ of the term of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ with the linear occurrence of u_{l-4} is a function of l, u_0 and $v_0, \ldots, v_4$ of the form

$$\begin{aligned}A_{(\mathbf{0}_{l-5}, 1, \mathbf{0}_4)}(v_0, \mathbf{v}_l)(u_0) = {} & -(v_4 l^4 + (-10v_4 - 4u_0v_1v_3)l^3 + \\ & (35v_4 + 36u_0v_1v_3 - 6u_0v_2^2 + 6u_0^2v_1^2v_2)l^2 + \\ & (-50v_4 - 112u_0v_1v_3 + 42u_0v_2^2 - 30u_0^2v_1^2v_2 - 4u_0^3v_1^4)l + \\ & (24 - 6u_0v_0)v_4 + (8u_0^2v_0 + 128u_0)v_1v_3 + (3u_0^2v_0 - 72u_0)v_2^2 + \\ & (24u_0^2 - 6u_0^3v_0)v_1^2v_2 + (u_0^4v_0 + 16u_0^3)v_1^4)/(24l - 96)\end{aligned}$$

Proof. We use the inductive definition of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ to obtain a recursive formula for the coefficient $A_{(\mathbf{0}_{l-m-1}, 1, \mathbf{0}^m)}(v_0, \mathbf{v}_l)(u_0)$ for the all the monomials where $m \geq 0$ and u_{l-m} occurs linearly in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$. The recursion in this formula runs over m.

Let $m = 0$. We use the fact that u_l does not occur in $E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa})$ for any $1 \leq \kappa \leq l$. Then using the definition of E_l one obtains the formula for $A_{\mathbf{0}^{l-1}1}$, by setting λ to l in the products of the form $\binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda}$ of that definition:

$$\begin{aligned} A_{\mathbf{0}_{l-1}1}(v_0, \mathbf{v}_l)(u_0) &= -(1/l) \sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0) \binom{\kappa}{l} \cdot v_{\kappa-l} \\ &\overset{\kappa \geq l \ ,\kappa \leq l \Rightarrow \kappa = l}{=} (-1/l) A_{()}()(u_0) \cdot v_0 \\ &= -v_0/l. \end{aligned}$$

Let $m > 0$. One sets $\lambda := l - m$ for the products of $u_{l-m} v_{\kappa-(l-m)}$ and the terms of $E_{l-\kappa}$ with no occurrences of u_{l-m}, and $\lambda := 0$ for the products of $u_0 v_\kappa$ and the terms of $E_{l-\kappa}$ where $u_{l-m} = u_{(l-\kappa)-(m-\kappa)}$ occurs linearly. We obtain the following equalities:

$$\begin{aligned} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_m}(v_0, \mathbf{v}_l)(u_0) = & -(1/l) \sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0) \binom{\kappa}{l-m} v_{\kappa-(l-m)} \\ & -(1/l) \sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_{l-\kappa-(l-m)}}(v_0, \mathbf{v}_{l-\kappa})(u_0) u_0 v_\kappa \\ & = -(1/l) \sum_{\kappa=l-m}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0) \binom{\kappa}{l-m} v_{\kappa-(l-m)} \\ & -(1/l) \sum_{\kappa=1}^{m} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_{l-\kappa-(l-m)}}(v_0, \mathbf{v}_{l-\kappa})(u_0) u_0 v_\kappa \\ & \overset{k := l-\kappa}{=} -(1/l) \sum_{k=0}^{m} A_{\mathbf{0}_k}(\mathbf{v}_k)(u_0) \binom{l-k}{l-m} v_{m-k} \\ & -(1/l) \sum_{\kappa=1}^{m} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_{l-\kappa-(l-m)}}(v_0, \mathbf{v}_{l-\kappa})(u_0) u_0 v_\kappa \end{aligned}$$

We have encoded this recursive over m definition for $A_{\mathbf{0}_{l-m-1}1\mathbf{0}_m}(v_0, \mathbf{v}_l)(u_0)$ as a program in the computer algebra system. To obtain the statement of the lemma one runs this program for $m = 4$. □

A.5. Tables of the coefficients for the analysis of ADE's with variable polynomial coefficients.

This subsection provides detailed technical information for Section 6. The expressions for E_l^*, $A_{\mathbf{i}_l}^*$ and $B_{l,m}^*$ for $0 \leq l \leq 3$ given here are used in the proof of Theorem 6. They are obtained by programming the corresponding recursive definitions in the computer algebra system Maxima. One can download the corresponding script `variableCoefficients` at `http://resourceanalysis.cs.ru.nl/#Algebraic Difference Equations`.

The expressions for E_l^*:

E_l^*	expression
$E_0^*(w_0, v_0, u_0)$	w_0
$E_1^*(w_0, w_1, v_0, v_1, u_0, u_1)$	$w_1 - u_0 w_0 v_1 - v_0 w_0 u_1$
$E_2^*(\mathbf{w}_2, v_0, \mathbf{v}_2, u_0, \mathbf{u}_2)$	$w_2 - (u_0 w_0 v_2)/2 - (v_0 w_0 u_2)/2 - u_0 v_1 w_1 - v_0 u_1 w_1 +$
	$(u_0^2 w_0 v_1^2)/2 + u_0 v_0 w_0 u_1 v_1 - w_0 u_1 v_1 + (v_0^2 w_0 u_1^2)/2$
$E_3^*(\mathbf{w}_3, v_0, \mathbf{v}_3, u_0, \mathbf{u}_3)$	$w_3 - (u_0 w_0 v_3)/3 - (v_0 w_0 u_3)/3 - u_0 v_1 w_2 -$
	$v_0 u_1 w_2 - (u_0 w_1 v_2)/2 + (u_0^2 w_0 v_1 v_2)/2 + (u_0 v_0 w_0 u_1 u_2)/2 -$
	$w_0 u_1 v_2 - (v_0 w_1 u_2)/2 +$
	$(u_0 v_0 w_0 v_1 u_2)/2 - w_0 v_1 u_2 + (v_0^2 w_0 u_1 u_2)/2 +$
	$(u_0^2 v_1^2 w_1)/2 + u_0 v_0 u_1 v_1 w_1 - u_1 v_1 w_1 + (v_0^2 u_1^2 w_1)/2 -$
	$(u_0^3 w_0 v_1^3)/6 - (u_0^2 v_0 w_0 u_1 v_1^2)/2 + u_0 w_0 u_1 v_1^2 -$
	$(u_0 v_0^2 w_0 u_1^2 v_1)/2 + v_0 w_0 u_1^2 v_1 - (v_0^3 w_0 u_1^3)/6$

(A.5)

Expressions for $A_{\mathbf{0}_l}^*$:

$A_{\mathbf{0}_l}^*(\mathbf{w}_l, \mathbf{v}_l)(u_0)$	expression
$A_{()}^*(w_0)(u_0)$	w_0
$A_0^*(w_0, w_1, v_1)(u_0)$	$w_1 - u_0 w_0 v_1$
$A_{00}^*(\mathbf{w}_2, \mathbf{v}_2)(u_0)$	$w_2 - (u_0 w_0 v_2)/2 - u_0 v_1 w_1 + (u_0^2 w_0 v_1^2)/2$

(A.6)

Expressions for $B_{l,m}^*$:

$B_{l,m}^*(\mathbf{w}_l, \mathbf{v}_l)$	expression
$B_{()}^*(w_0)$	w_0
$B_{1,0}^*(w_0, w_1, v_1)$	w_1
$B_{1,1}^*(w_0, w_1, v_1)$	$-w_0 v_1$
$B_{2,0}^*(\mathbf{w}_2, \mathbf{v}_2)$	w_2
$B_{2,1}^*(\mathbf{w}_2, \mathbf{v}_2)$	$-(w_0 v_2)/2 - v_1 w_1$
$B_{2,2}^*(\mathbf{w}_2, \mathbf{v}_2)$	$(w_0 v_1^2)/2$

(A.7)

Expressions for $A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$, where $\mathbf{i}_l \neq \mathbf{0}_l$:

$A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$	expression	presentation via $B^*_{l,m}$
$A^*_1(w_0, w_1, v_0, v_1)(u_0)$	$-v_0 w_0$	$-v_0 B_{0,0}(w_0)$
$A^*_{10}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$	$-v_0 w_1 + u_0 v_0 w_0 v_1 - w_0 v_1$	$(1 - u_0 v_0) B_{1,1}(w_0, w_1, v_1) -$ $v_0 B_{1,0}(w_0, w_1, v_1)$
$A^*_{20}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$	$(v_0^2 w_0)/2$	$(v_0^2)/2 B_{0,0}(w_0)$
$A^*_{01}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$	$-(v_0 w_0)/2$	$-(v_0/2) B_{0,0}(w_0)$
$A^*_{100}(\mathbf{w}_3, v_0, \mathbf{v}_3, \mathbf{u}_3)(u_0)$	$-v_0 w_2 + (u_0 v_0 w_0 v_2)/2 - w_0 v_2 -$ $u_0 v_0 v_1 w_1 - v_1 w_1 -$ $(u_0^2 v_0 w_0 v_1^2)/2 + u_0 w_0 v_1^2$	$(-v_0) B_{2,0}(\mathbf{w}_2, \mathbf{v}_2) +$ $(2u_0 - u_0^2 v_0) B_{2,2}(\mathbf{w}_2, \mathbf{v}_2) +$ $(1 - u_0 v_0) B_{2,1}(\mathbf{w}_2, \mathbf{v}_2) -$ $(w_0 v_2)/2$

(A.8)

A.6. Connection between Diophantine equations and ADE's

We consider a solution tuple $a_0, \ldots, a_m \in \mathbb{A}$ of a Diophantine equation

$$F(x_0, \ldots, x_m) = 0 \tag{A.9}$$

as the coefficients of the polynomial $P(x) = a_m x^m + \cdots + a_1 x + a_0$. Also, we consider an $(m+1) \times (m+1)$ linear system of symbolic equations w.r.t. $a_m, \ldots, a_0$:

$$\begin{aligned} &a_m x^m + \cdots + a_1 x + a_0 &&= P(x) \\ &a_m (x-1)^m + \cdots + a_1 (x-1) + a_0 &&= P(x-1) \\ &\cdots \\ &a_m (x-m)^m + \cdots + a_1 (x-m) + a_0 &&= P(x-m). \end{aligned} \tag{A.10}$$

Using Kramer's rule one obtains the rational symbolic algebraic expressions of the form $a_i = \Delta_i(x, P)/\Delta(x)$, where :

$$\Delta(x) = \begin{vmatrix} x^m & x^{m-1} & \ldots & x & 1 \\ (x-1)^m & (x-1)^{m-1} & \ldots & x-1 & 1 \\ \cdots \\ (x-m)^m & (x-m)^{m-1} & \ldots & x-m & 1 \end{vmatrix} \tag{A.11}$$

is the determinant of the system, and

$$\Delta_i(x,P) = \begin{vmatrix} x^m & \dots\ x^{m-(i-1)} & P(x) & x^{m-(i+1)} & \dots\ 1 \\ (x-1)^m & \dots\ (x-1)^{m-(i-1)} & P(x-1) & (x-1)^{m-(i+1)} & \dots\ 1 \\ \dots \\ (x-m)^m & \dots\ (x-m)^{m-(i-1)} & P(x-m) & (x-m)^{m-(i+1)} & \dots\ 1 \end{vmatrix} \quad \text{(A.12)}$$

is the minor corresponding to the variable a_i. Now we substitute these expressions into the original Diophantine equation and obtain the following rational algebraic equation

$$F\big(\tfrac{\Delta_0(x,P)}{\Delta(x)}, \dots, \tfrac{\Delta_m(x,P)}{\Delta(x)}\big) = 0 \quad \text{(A.13)}$$

which after the multiplication of both parts by the symbolic denominator $\Delta^D(x)$, where D is the total degree of the polynomial F, yields the ADE of the form

$$G_F(x)\big(P(x), P(x-1), \dots, P(x-m)\big) = 0. \quad \text{(A.14)}$$

Lemma 26. Diophantine equation (A.9) has an integer solution tuple if and only if the derived ADE (A.14) has a polynomial solution in $\mathbb{Z}[x]$.

Proof. Let the Diophantine equation have an integer solution tuple $a_0, \dots, a_m$. Introduce the polynomial $P(x) := a_m x^m + \dots + a_1 x + a_0$. It is a routine to show that by the construction of ADE's (A.14) this polynomial is its solution. Indeed, from the symbolic equalities $a_i = \Delta_i(x,P)/\Delta(x)$, where P is considered as a symbol, it follows that equation (A.13) holds for $P(x)$, but this equation is equivalent to ADE (A.14), since $\Delta(x) \neq 0$ because it is the Vandermonde determinant with the entries $1, x, x-1, \dots x-m$. Now, let ADE (A.14) has a polynomial solution $P(x) := a_m x^m + \dots + a_1 x + a_0$. Then, again, it is easy to check that $(a_0, \dots, a_m)$ solves $F(y_0, \dots, y_m) = 0$. Indeed, ADE (A.14) is equivalent to rational equation (A.13) and since $\Delta_i(x,P)/\Delta(x) = a_i$ then $F(a_0, \dots, a_m) = 0$ as well. □

Theorem 7. There is no algorithm that for any m and any ADE with integer coefficients decides if it has an integer polynomial solutions of degree at most m or not.

Proof. The theorem follows from Lemma 26 and the fact that the problem of the existence of the roots of Diophantine equations is undecidable (Davis, 1973). In particular if an undecidable Diophantine equation $F(a_0, \dots, a_m)$ is given then there is no decision procedure that for the corresponding equation $G_F\big(P(x), P(x-1), \dots, P(x-m), x\big) = 0$ decides if it has an integer polynomial solution of degree at most m or not. □

Recall that the decidability of the Diophantine problem in the field $\mathbb{Q}$ of rational numbers is still open.

A.7. Undecidability of the positive existential theory for polynomial rings with the difference operator

Let $\mathbb{K}$ be a number field. In Section 7 we have shown that knowing the degree of a possible polynomial solution of an ADE does not guarantee that one can find this solution or prove its absence if the first-order theory of $\mathbb{K}$ is undecidable. In this section we will show that whichever the first-order theory of $\mathbb{K}$ is, finding polynomial solutions of systems of equations involving ADE's, is, in general, an undecidable problem.

Recall that a *positive existential theory* in a language L is a the set of all first-order existential sentences, containing only equations, in the language L which are true in $\mathbb{K}[x]$ (Pheidas and Zahidi, 2000). Let Δ denote the difference operator $\Delta(P)(x) := P(x) - P(x-1)$.

The following construction is used to show the representation of integer numbers via linear ADE's with variable coefficients. We consider an n-parametric family of the rising-factorial polynomials of the form

$$P_n(x) := (x+1)\cdots(x+n). \tag{A.15}$$

When the number n is fixed, for the polynomial $P_n(x)$ the following equation holds:

$$\big(P_n(x) - P_n(x-1)\big)(x+n) = nP_n(x) \tag{A.16}$$

which is easy to check by routine calculations:

$$\begin{aligned} P_n(x) - P_n(x-1) = {} & (x+1)\cdots(x+n) - x\cdots(x+n-1) = \\ & (x+1)\cdots(x+n-1)\big((x+n)-x\big) = \\ & \frac{P_n(x)}{x+n}n. \end{aligned} \tag{A.17}$$

Equation (A.16) gives an idea for the following definition of the integer number.

Lemma 27. A number $b \in \mathbb{K}$ is integer if and only if there is a non-zero polynomial solution $P \in \mathbb{K}[x]$ for the difference equation

$$\big(P(x) - P(x-1)\big)(x+b) = bP(x). \tag{A.18}$$

Moreover, b is the degree of this polynomial solution.

Proof. Let equation (A.18) does have a polynomial solution $P \in \mathbb{K}[x]$ and let $P(x) := a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$. The coefficients of x^d and of x^{d-1} in $P(x-1)$ are a_d and $(-a_d d + a_{d-1})$ respectively. This implies that the coefficient of x^d in $P(x) - P(x-1)$ is $a_d - a_d = 0$ and the coefficient of x^{d-1} is $a_{d-1} - (-a_d d + a_{d-1}) = a_d d$. Therefore the coefficient of x^d on the left-hand side of equation (A.18) is $a_d d$. Trivially the coefficient of x^d on the right-hand side is ba_d. Therefore one has $a_d d = ba_d$ and therefore $d = b$. So, b is an integer number, and, moreover, it is equal to the degree of the polynomial solution.

Now, let $b = n$ be an integer number. Then by the definition (A.15) and equation (A.16) one has that the rising-factorial polynomial of degree $b = n$ is a polynomial solution of equation (A.18). $\square$

Now, we want to obtain the equivalent definition of b being an integer number in the positive existential theory for $\mathbb{K}[x]$ with the difference operator Δ, that is we want to exclude the universal quantifier in the formula $\exists\ P.\forall x.\big((x+b)\Delta[P](x) = bP(x)\big)$. Let $\mathbb{id}$ denote the identity polynomial, which sends x to itself. Then formula for defining b as integer number above can be written as $\exists\ P.\forall x.\big(\mathbb{id}(x) + b\big)\Delta[P](x) = bP(x)$ and the following statement holds:

$$b \in Z^+ \text{ if and only if } \exists\ P\ H.(\mathbb{Id} + b)\Delta[P] = bP \wedge PH = 1, \tag{A.19}$$

where $\exists\ H.PH = 1$ is equivalent to "P is a non-zero polynomial." Now it is a routine to prove the following theorem.

Theorem 8. The positive existential theory of $\mathbb{K}[x]$ in the language L_Δ is undecidable.

Proof. The proof mimics the proof of undecidability of the positive existential theory of complex rational functions in the language augmented with the derivative operator, (Pheidas and Zahidi, 1999).

Due to Lemma 27 and equation (A.19), a Diophantine equation $F(y_1, \ldots, y_m) = 0$ has an integer solution if and only if the formula $\exists\ b_1, \ldots, b_m, P_1, \ldots, P_m.\ F(b_1, \ldots, b_m) = 0 \wedge (\mathbb{Id}+b_1)\Delta[P] = b_1 P \wedge \cdots \wedge (\mathbb{Id}+b_m)\Delta[P] = b_m P \wedge P_1 H_1 = 1 \wedge P_m H_m = 1$ is true in $\mathbb{K}[x]$. Since the solvability of the Diophantine problem for integers reduces to the decidability of the positive theory of $\mathbb{K}[x]$ in the language L_Δ, the latter theory is undecidable. □

A.8. The influence of G_0 on the existence of an upper bound of the degree of a polynomial solution

Since the polynomial $G_0(x)$ is not involved in the main work behind the presented approach, it may be tempting to remove it from the formulation of the main results and to use assumptions like "without loss of generality assume that G_0 is the zero polynomial".

However, we decided to keep $G_0(x)$ in the formulations because it does influence the existence of an upper bound of the degree of a polynomial solution of an ADE. An example is given by the following pair of the ADE's that differ only by $G_0(x)$, with $G_0(x) \equiv 0$ for the first one, and $G_0(x) \equiv -1$ for the second one:

$$\begin{aligned} &P(x)P(x-2)P(x-3) - 2P(x-1)P(x-1)P(x-3)+ \\ &P(x-1)P(x-2)P(x-2) + P(x)P(x-1)P(x-3)- \\ &2P(x)P(x-2)P(x-2) + P(x-1)P(x-1)P(x-2) = 0, \end{aligned} \tag{A.20}$$

$$\begin{aligned} &P(x)P(x-2)P(x-3) - 2P(x-1)P(x-1)P(x-3)+ \\ &P(x-1)P(x-2)P(x-2) + P(x)P(x-1)P(x-3)- \\ &2P(x)P(x-2)P(x-2) + P(x-1)P(x-1)P(x-2) = 1. \end{aligned} \tag{A.21}$$

There is no an upper bound for the degree of a polynomial solution for the first ADE because for any real a the corresponding falling factorial of the form $P_n(x) = (x-a)(x-a-1)\cdots\big(x-a-(n-1)\big)$ is its solution, see (Shkaravska and van Eekelen, 2014). The second equation has polynomial solutions of degree at most 2. Below we discover why this is the case.

Let $\mathbb{K}$ be a number field.

Lemma 28. For $l \geq 5$ the coefficient of the linear occurrence of u_{l-4} in $S_l(u_0, \mathbf{u}_l)$ for equations (A.20) and (A.21) is $K(u_0, l) = -u_0 l^2 + 9u_0 l - 20u_0$. (Note that this polynomial is equal to zero in $l = 4, 5$.)

Proof. Recall that this coefficient is equal to the sum $\sum_{\mathbf{t}_3\in T}\alpha_{\mathbf{t}_3}A_{\mathbf{0}_{l-5}1\mathbf{0}_4}(u_0)(3,\mathbf{p}_l(\mathbf{t}_3))$. The expression for $K(u_0,l)$ is calculated straightforwardly in the computer algebra system as this sum, using Lemma 25. □

Lemma 29. If $d\geq 3$ then, up to the leading coefficient, the only solutions of equation (A.20) are rising factorials of the form $(x+a)(x+a+1)\cdots\big(x+a+(d-1)\big)$.

Proof. We fix arbitrary $d\geq 3$ and some solution $P(x)=x^d+a_{d-1}x^{d-1}+\cdots+a_1x+a_0$ of degree d. We are going to show all the values $p_2(\mathbf{r}_d),\ldots,p_d(\mathbf{r}_d)$ of the power-sum symmetric polynomials at the roots of the solution are defined uniquely as functions of $p_1(\mathbf{r}_d)$.

Indeed, for equation (A.20) the polynomials $S_0(u_0),S_1(u_0,0),S_2(u_0,0,0),S_3(u_0,\mathbf{0}_3)$ are all equal to the zero polynomial, and therefore one can apply Lemmata 20, 21, 22,23 and Lemma 28. Therefore for each $l\geq 6$ there is the nonzero coefficients $K(d,l)=d(-l^2+9l-20)$ and an algebraic expression $M(l,d,\mathbf{u}_{l-5})$ such that $S_l(d,\mathbf{u}_l)=K(d,l)u_{l-4}+M(l,d,\mathbf{u}_{l-5})$.

Since $d\geq 3$ one has that $3d-(d+4)\geq 1$, and therefore for all $6\leq l\leq d+4$ the values $S_l\big(d,\mathbf{p}_l(\mathbf{r}_l)\big)=0$. Together with the presentation above this implies that

$$p_{l-4}(\mathbf{r}_d)=-\frac{M\big(l,d,\mathbf{p}_{l-5}(\mathbf{r}_d)\big)}{K(d,l)} \tag{A.22}$$

By induction from this follows that $p_{l-4}(\mathbf{r}_d)$ is defined uniquely via its predecessors $p_1(\mathbf{r}_d),\ldots,p_{l-5}(\mathbf{r}_d)$ for $l=6,\ldots,d+4$. Therefore all the coefficients $a_l=(-1)^le_l(\mathbf{r}_d)$ of a polynomial solution are defined uniquely via $p_1(\mathbf{r}_d)$ using the Newton-Girard identities and the identity (A.22) above. For the given a_{d-1} one can find such a that

$$a_{d-1}=-p_1(\mathbf{r}_d)=-\big(a+(a+1)+\cdots+(a+d-1)\big). \tag{A.23}$$

The rising factorial corresponding to this $a\in\mathbb{K}$ solves equation (A.20) and therefore the power-sums at its roots satisfy the identity (A.22) as well. Therefore $P(x)$ coincides with the rising factorial for this a. □

Now we can prove the main statement of this section.

Lemma 30. Equation (A.21) does not have solutions of degree $d\geq 3$.

Proof. The proof is similar to the proof of Lemma 29. Let us assume the opposite, that is for equation (A.21) there exists a polynomial solution $P(x)=x^d+a_{d-1}x^{d-1}+\cdots+a_1x+a_0$ of degree $d\geq 3$. We note that for both equations (A.20) and (A.21) the functions $S_l(u_0,\mathbf{u}_l)$ are the same and, moreover, the coefficients of x^{3d-l} must vanish for all $6\leq l\leq d+4$ if we speak about their solutions of degree $d\geq 3$. This implies that for the roots of solutions of both equations identities (A.22) hold. For the given a_{d-1} one can find such a that equality (A.23) holds. Therefore by induction on $l=1,\ldots,d$ we obtain that the corresponding coefficients of the polynomial $P(x)$ and of the rising factorial for this a are equal. Therefore both polynomials are equal and $P(x)$ cannot solve equation (A.21), since the rising factorial solves equation (A.20). □

Lemma 31. Equation (A.21) does have solutions of degree $d\leq 2$.

Proof. The solutions of degree $d \leq 2$ are obtained via the method of unknown coefficients using the computer algebra system. For instance, it can be shown that for $d = 2$ the normalised solutions are of the form $P(x) = x^2 + a_1 x + a_0$ where $a_0 = (2a_1^2 - 1)/8$. $\square$